MODULAR COMPACTIFICATION OF MODULI OF K3 SURFACES OF DEGREE 2

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Contents

1. Introduction

This paper is concerned with compactifications of the moduli spaces F_{2d} of polarized K3 surfaces (S, L), $L^2 = 2d$, and more narrowly with the case $2d = 2$. It is well known that F_{2d} is a quasiprojective variety of dimension 19 and that over $\mathbb{C}$ it can be written as a quotient $\mathcal{D}/G$ of a symmetric Hermitian domain of type IV by a discrete arithmetic group. There are several approaches to compactifying F_{2d}:

(1) The first one is specific to the quotients of the form $\mathcal{D}/G$, where the fact that it happens to be a moduli space is irrelevant. Namely, for any space $\mathcal{D}/G$ there is a Baily-Borel compactification [BB66] $\overline{\mathcal{D}/G}^{\rm BB}$ with a fairly small boundary (for type IV domains only curves and points). Further, there exist infinitely many toroidal compactifications $\overline{\mathcal{D}/G}^{\tau}$ of Mumford et al [AMRT75]. These depend on choices of compatible fans for the cusps in the Baily-Borel compactification. The boundary of each $\overline{\mathcal{D}/G}^{\tau}$ is a divisor, and they all map to $\overline{\mathcal{D}/G}^{\rm BB}$. A related compactification, in a sense a mixture of the above two, is due to Looijenga [Loo86, Loo03].

(2) The second approach is to use Geometric Invariant Theory. For K3 surfaces of degrees 2 and 4 it was pioneered by Shah [Sha80, Sha81].

(3) The third approach is to compactify F_{2d} functorially as a moduli space by adding some "stable surfaces" on the boundary, in a way similar to the Deligne-Mumford compactification $\overline{M}_g$ for the moduli space of curves. The general theory [KSB88], [Ale96a, Ale96b, Ale06], [Kol15], sometimes going by the moniker KSBA, says that the moduli functor of pairs $(S, \sum d_i D_i)$ with semi log canonical singularities and ample $K_S + \sum d_i D_i$ has a proper moduli space. Although there are some remaining technical questions in the most general case, in the special case of K3

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surface pairs $(S, \epsilon D)$ with a small coefficient $0 < \epsilon \ll 1$ and an ample divisor D the theory is complete.

This provides e.g. a compactification for the moduli space P_{2d} of pairs $(S, \epsilon D)$ of K3 surfaces with the additional data of a divisor $D \in |L|$ in the polarization class. This space has dimension $20+d$ and it is fibered over F_{2d} into (quotients of) $\mathbb{P}^{d+1}$s. The space $\overline{P}_2$ was computed by Laza [Laz12].

To compactify the original space F_{2d} one has to make some intrinsic choice of a divisor $D \in |NL|$ that would depend only on the surface S itself, for some fixed multiple N. There are many natural choices for such an intrinsic D, leading to different compactifications. In our particular case, every K3 surface S of degree 2 comes with an involution and a 2-to-1 map $\pi\colon S \to X$ to a $\mathbb{P}^2$ or $\mathbb{F}_4^0$ (the Hirzebruch surface $\mathbb{F}_4$ with the (-4)-section contracted), and one can take for $D \in |3L|$ the ramification curve of π. This is probably the smallest possible choice, and it is the one we make.

It is easy to see that given a double cover $\pi\colon S \to X$, the pair $(S, 2\epsilon D)$ is stable iff the pair $\big(X, (\frac{1}{2}+\epsilon)B\big)$ is stable, where $B \sim -2K_X$ is the branch divisor on the base.

Thus, the moduli compactification $\overline{F}_2$ is essentially the same as the compactification $\overline{M}(\mathbb{P}^2, d)$ of the moduli space of planar degree d pairs $\big(\mathbb{P}^2, (\frac{3}{d}+\epsilon)C_d\big)$ constructed by Hacking in [Hac04], in the special case of sextic curves. Hacking's results are very complete for $d = 4, 5$ and somewhat complete for $3 \nmid d$. The expanded version [Hac01] contains some partial results for the $d = 6$ case which are unfortunately not enough to describe this space in detail. We supplement the results of that work here.

A very interesting question is whether one can combine approaches (1) and (3):

Question 1.1. Does one of the infinitely many toroidal compactifications $\overline{F}_{2d}^{\tau}$ have a special moduli meaning? Does it come with a family of stable K3 surface pairs?

The answer to this question is known to be "yes" in the closely related but easier case of principally polarized abelian varieties whose moduli space is a quotient $A_g = \mathcal{H}_g / \operatorname{Sp}(2g, \mathbb{Z})$ of a symmetric domain of Siegel type. In that case one finds [Ale02] that the normalization of the main component of the moduli compactification $\overline{A}_g$ coincides with the toroidal compactification $\overline{A}_g^{\tau}$ for a special fan called the 2nd Voronoi fan. (In the definition of the moduli functor, principally polarized abelian varieties (A, λ) are replaced by pairs $(Y, \epsilon\Theta)$ consisting of an abelian torsor $A \curvearrowright Y$ together with a theta divisor; such pairs have the same moduli space A_g.) So the above question can be put in the following way:

Question 1.2. Is there a substitute for the 2nd Voronoi fan in the case of polarized K3 surfaces?

The main aim of this paper is to relate a particular toroidal compactification $\overline{F}_2^{\text{refl}}$ with the moduli compactification $\overline{F}_2$ by stable pairs $(S, \epsilon D)$ for the above choice of $D \in |3L|$ as the ramification divisor, i.e. $\overline{F}_2 = \overline{M}(\mathbb{P}^2, 6)$. The fan that we consider is the reflection fan τ^{refl} cut out by the hyperplanes perpendicular to the (-2)-vectors in the relevant lattice. Our main result is as follows:

Theorem 1.3. *The strata of $\overline{F}_2$ are in a bijection with the strata of $\overline{F}_2^{\text{refl}}$ modulo a certain natural equivalence relation defined in* (4.6).

In light of this we make the following conjecture:

Conjecture 1.4. *There exists a universal family of stable pairs $(\mathcal{X}, \epsilon\mathcal{D})$ over the toroidal compactification $\overline{F}^{\text{refl}}$, which defines a morphism $\overline{F}_2^{\text{refl}} \to \overline{F}_2$.*

Remark 1.5. The space $\overline{F}_2$ itself is definitely *not* a toroidal compactification of F_2 because a particular stratum correponding to the $2\widetilde{E}_8\widetilde{A}_1$ type II degenerations has codimension 2 in $\overline{F}_2$, but it should be a divisor in any toroidal compactification. However, the positive-dimensional fibers of $\overline{F}_2^{\text{refl}} \to \overline{F}_2$ are small, of dimension ≤ 3, and there are few of them. These fibers are described by the equivalence relation (4.6), as in Theorem 1.3.

The plan of the paper is as follows. In Section 2 we briefly review the relevant combinatorics of toroidal compactifications as it relates to our case. We also describe the reflection fan τ^{refl} in detail. It turns out that cones in this fan are conveniently described by parabolic and elliptic subdiagrams of a certain weighted graph called the Vinberg diagram Γ_{Vin}.

In Section 3 we briefly review the theory of moduli compactifications via stable pairs, explain how it applies to the case of K3 surfaces of degree 2, and recall Hacking's results on the compactification $\overline{M}\big(\mathbb{P}^2, (\frac12+\epsilon)B_6\big)$ of planar sextics.

In Section 4 we describe all stable limits of K3 pairs $(S, \epsilon D)$, i.e. of pairs $\big(\mathbb{P}^2, (\frac12 + \epsilon B)\big)$ and $\big(\mathbb{F}_4^0, (\frac12+\epsilon B)\big)$, and find that the types of limits are in a bijection with the cones of τ^{refl} modulo a certain natural equivalence relation defined in (4.6).

In Section 5 we study a closely related question. We classify maximally log canonical sextics in the plane, by which we mean that $(\mathbb{P}^2, \frac12 B)$ is lc but the pair $\big(\mathbb{P}^2, (\frac12+\epsilon)B\big)$ for $\epsilon > 0$ is not lc. Our theorem in this case is:

Theorem 1.6. *There is a bijection between the types of maximally log canonical sextics in the plane and the hyperbolic subdiagrams of a certain graph Γ_{21}, obtained by removing three central vertices from the Vinberg diagram Γ_{Vin}.*

In a sense, this theorem is dual to a description of cones in τ^{refl}. Dualizing hyperbolic diagrams gives parabolic and elliptic subdiagrams of Γ_{21} and Γ_{Vin}.

In Section 6 we prove a result (Theorem 6.13) that gives sufficient conditions for a family to satisfy Conjecture 1.4, and present a construction of a family that comes very close to satisfying these conditions.

Throughout the paper, we work over the field $\mathbb{C}$ of complex numbers.

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2. Toroidal compactifications of F_2

2.1. Baily-Borel and toroidal compactifications of F_{2d}. The general references for this section are [BB66, AMRT75, ast85]; see also [Sca87, Kon93, Loo03].

The cohomology group $H^2(S,\mathbb{Z})$ of a smooth K3 surface with the intersection pairing is isomorphic to the unique even unimodular lattice $\mathrm{II}_{3,19} = \mathrm{E}_8^{\oplus 2} \oplus \mathrm{U}^{\oplus 3}$, where E_8 and U are the standard even unimodular lattices of signatures $(0,8)$ and $(1,1)$. Let $h \in \mathrm{II}_{3,19}$ be a primitive vector with $h^2 = 2d > 0$, and let $L_{2d} := h^\perp \simeq$

$E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus \langle -2d\rangle$, a lattice of signature $(2,19)$. The domain $\mathcal{D}_{2d}$ is defined as either of the two connected components of the set

$$\{[\omega] \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) \mid \omega.\omega = 0,\ \omega.\bar{\omega} > 0\}.$$

Let G_{2d} be the subgroup of the isometry group $O(\mathrm{II}_{3,19})$ that fixes h and the connected component $\mathcal{D}_{2d}$. This is a subgroup of finite index in $O(L_{2d})$. Then the Global Torelli Theorem says that there is a bijection between

(1) the points of the quotient $\mathcal{D}/G$ and
(2) the isomorphism classes of pseudo-polarized K3 surfaces $(\widehat{X}, \widehat{L})$ with a big and nef line bundle $\widehat{L}$ such that $\widehat{L}$ is primitive in $\operatorname{Pic}\widehat{X}$ and $\widehat{L}^2 = 2d$, or
(2′) equivalently, with the isomorphism classes of polarized K3 surfaces (X,L) that have Du Val (A_n, D_n, E_n) singularities, with a primitive ample line bundle L such that $L^2 = 2d$.

Given $(\widehat{X}, \widehat{L})$, the pair (X,L) is obtained by contracting the (-2)-curves $C \subset X$ such that $\widehat{L}.C = 0$. Given (X,L), the surface $\widehat{X}$ is the minimal resolution of singularities of X, and $\widehat{L}$ is the pullback of L.

Many authors prefer to work with the smooth K3 surfaces, and the coarse moduli space in the category of schemes for either moduli functor is D_{2d}/G_{2d}, which is a quasiprojective variety over $\mathbb{C}$. However, the moduli functor for smooth K3s is non-separated. Indeed, for a generic 1-parameter family $\mathcal{X} \to \mathcal{S}$ with a central fiber $\mathcal{X}_0$ acquiring a (-2)-curve, $\mathcal{X}$ can be flipped in C, giving a non-isomorphic over $\mathcal{S}$ family $\mathcal{X}^+ \to \mathcal{S}$. For this reason, it is preferable to work with the functor of polarized K3 surfaces (X,L) with ADE singularities whose moduli functor is separated.

The boundary of Baily-Borel compactification $\overline{F}_{2d}^{\mathrm{BB}}$ consists of curves (1-cusps) which intersect and self-intersect at points (0-cusps). The 0-cusps are in a bijection with primitive vectors $e \in L_{2d}$ such that $e^2 = 0$, modulo G_{2d}. Similarly, the 1-cusps are in a bijection with primitive isotropic sublattices $E \subset L_{2d}$ of rank 2, modulo G_{2d}. The 1-cusps (resp. 0-cusps) correspond to type II (resp. type III) degenerations of K3 surfaces.

For any 0-cusp e, let $N_{2d,e}$ be the lattice $e^\perp/e$ of signature $(1,18)$, and let $G_{2d,e}$ be the subgroup of G_{2d} fixing e. The data for a toroidal compactification $\overline{F}_{2d}^{\tau}$ is a collection of $G_{2d,e}$-equivariant fans $\{\tau_e\}$, one for each 0-cusp. (Recall that a fan is a locally finite collection of face-fitting rational polyhedral cones.) The fan τ_e is a fan in the $\mathbb{R}$-vector space with the lattice $N_{2d,e}$ and its support is the rational closure $\overline{C}_{2d,e}^{\mathbb{Q}}$ of the cone

$$C_{2d,e} = \{v \in N_{2d,e} \otimes \mathbb{R} \mid v^2 > 0\}$$

obtained by adding rational vectors v with $v^2 = 0$. The fan must be equivariant with respect to the $G_{2d,e}$-action, and there must be only finitely many cones modulo $G_{2d,e}$. The toroidal compactification $\overline{F}_{2d}^{\tau}$ is modeled on the $G_{2d,e}$-quotients of the "infinite toric varieties" corresponding to the fans τ_e.

For other symmetric domains, one may have to consider compatible fans for cusps of all dimensions. However, for the type IV domains the fans for the 1-cusps lie in 1-dimensional spaces, so these fans are unique and the compatibility is immediate.

2.2. **The degree 2 case.** We now specialize this construction to the degree $2d = 2$ case. Then $L_2 = E_8^{\oplus 2} \oplus U^2 \oplus A_1$, where $A_1 = \langle -2\rangle$. There is a unique 0-cusp and four 1-cusps. Consequently, the boundary $\overline{F}_2^{\rm BB} \setminus \overline{F}_2$ consists of four nodal curves meeting at a single point.

The lattice corresponding to the unique 0-cusp is $N_2 := N_{2,e} = E_8^{\oplus 2} \oplus U \oplus A_1$. We denote the dual lattice $M_2 = N_2^*$; this is an overlattice of N_2 of index 2. The corresponding group is $G_{2,e} = O(N_2)$.

Thus, the data for a toroidal compactification $\overline{F}_2^\tau$ is one $O(N_2)$-equivariant fan τ supported on $\overline{C}^{\mathbb{Q}}$. There is a natural choice for this fan: the reflection fan $\tau^{\rm refl}$, obtained by cutting $\overline{C}^{\mathbb{Q}}$ by hyperplanes orthogonal to roots $r \in N_2$, $r^2 = -2$. A fundamental domain of the lattice N_2 for this reflection action has 24 sides. The corresponding polyhedron in the hyperbolic space $\mathbb{H} = C/\mathbb{R}^+$ has finite volume.

Of course for an arbitrary hyperbolic lattice N the reflection group may not give a fan; this happens iff the fundamental polyhedron has infinite volume. Vinberg [Vin75] gave a constructive algorithm to determine whether the volume is finite. When it finishes successfully, Vinberg's algorithm produces a Vinberg diagram $\Gamma_{\rm Vin}$ (also called a Coxeter diagram) whose vertices correspond to the walls $r_i^\perp$ of a chosen fundamental domain $\delta = \{v \mid r_i.v \geq 0\}$, and edges indicate the angles between r_i's. The following is a general description of the faces of all dimensions of the fundamental domain from [Vin75]:

Theorem 2.1 (Vinberg)**.** *There is a bijection between faces F of the fundamental domain δ and certain subdiagrams $I \subset \operatorname{Ver}(\Gamma_{\rm Vin})$ defined by*

$$F \mapsto \{i \mid F \subset r_i^\perp\}, \qquad I \mapsto \cap_{i \in I}\, r_i^\perp \cap \delta.$$

The faces of δ are of two types:

1. *Rays $\mathbb{R}_{\geq 0} v$ with $v^2 = 0$. These are in a bijection with maximal subdiagrams of $\Gamma_{\rm Vin}$ whose every connected component is parabolic.*
2. *Other cones (including rays) which are in a bijection with elliptic subdiagrams of $\Gamma_{\rm Vin}$.*

The correspondence is order-reversing. The cone δ itself corresponds to $\emptyset$.

We call the above two types of cones *type II* and *type III* cones.

Corollary 2.2. *The cones of $\tau^{\rm refl}$ modulo $O(N_2)$ and the strata in the toroidal compactification $\overline{F}_2^{\rm refl}$ are in a bijection with the elliptic and maximal parabolic subdiagrams of $\Gamma_{\rm Vin}$ modulo $\operatorname{Aut}(\Gamma_{\rm Vin})$.*

For the lattice $N_2 = E_8^{\oplus 2} \oplus U \oplus A_1$, the Vinberg algorithm, proving that $\tau^{\rm refl}$ is a fan, was computed e.g. by Scattone in the last chapter of [Sca87], see also [Kon89, Nik83]. The Vinberg diagram $\Gamma_{\rm Vin}$ in this case has 24 vertices. We reproduce it in Figure 1. Each vertex corresponds to a vector r_i with $r_i^2 = -2$. There is no edge between the vertices r_i, r_j if $r_i r_j = 0$, a single edge if $r_i r_j = 1$, a double edge if $r_i r_j = 2$, and a dashed line if $r_i r_j = 6$. We denote the vectors $\vec{a}_i$, $\vec{b}_i$, $\vec{d}_i$ as shown. Note that $\operatorname{Aut}(\Gamma_{\rm Vin}) = S_3$.

The four maximal parabolic subdiagrams of $\Gamma_{\rm Vin}$, also found in [Sca87], are shown in Figures 2a–2d. Connected components of these diagrams are extended Dynkin $\widetilde{A}\widetilde{D}\widetilde{E}$ diagrams. Connected components of elliptic subdiagrams are ordinary Dynkin ADE diagrams. Each of the maximal elliptic subdiagrams has 18 vertices. Some of these are A_{18}, D_{18}, $3E_6$, etc.

By construction, the toroidal compactification $\overline{F}^{\rm refl}$ is modeled on a single ordinary 19-dimensional affine toric variety $U^{\rm refl}$ which we now explicitly define.

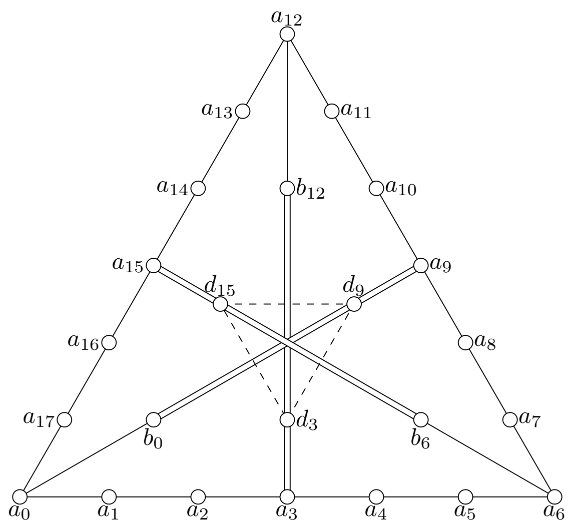
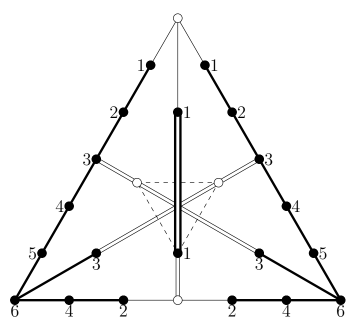
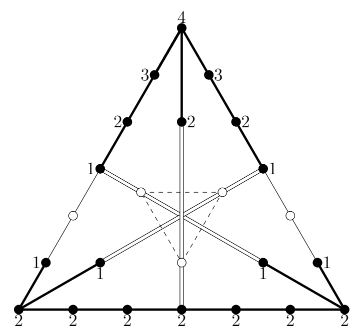


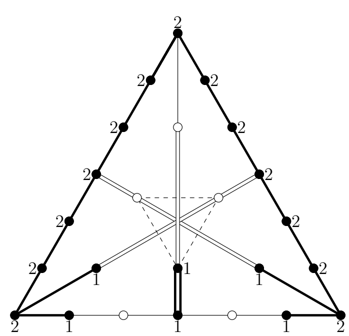
FIGURE 1. Vinberg diagram Γ_{Vin}



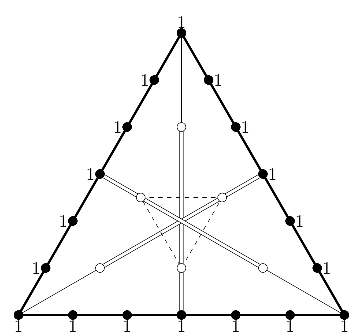
(A) $2\widetilde{E}_8\widetilde{A}_1$



(B) $\widetilde{D}_{10}\widetilde{E}_7$



(C) $\widetilde{D}_{16}\widetilde{A}_1$



(D) $\widetilde{A}_{17}$

FIGURE 2. Maximal parabolic subdiagrams of Γ_{Vin}

Definition 2.3. In the lattice M_2 of monomials, consider the rational polyhedral cone $\check\sigma^{\rm refl}$ generated by the integral vectors $\vec a_i$ $(i = 0, \dots 17)$, $\vec b_i$ $(i = 0, 6, 12)$, and $\vec d'_i = \frac{1}{2}\vec d_i$ $(i = 3, 9, 15)$. We denote by $\sigma^{\rm refl}$ its dual cone in N_2.

The affine toric variety corresponding to $\check\sigma^{\rm refl}$ and $\sigma^{\rm refl}$ is denoted by $U^{\rm refl}$.

Lemma 2.4. *The maximal-dimensional cone $\sigma^{\rm refl}$ has 10 rays of type II – 4 up to $O(N_2)$, and 525 rays of type III – 99 up to $O(N_2)$.*

Up to $O(N_2)$, each ray is uniquely determined by its label except for the cases $A_{13}A_4A_1$, $A_{17}A_1$, $A_9 2A_4A_1$, $A_9E_8A_1$, $D_{10}E_7A_1$, for which there are two rays.

Proof. A direct computation. □

2.3. **Relations between vectors $\vec a_i$, $\vec b_i$, $\vec d_i$.** Let $\Gamma = \sqcup \Gamma_s$ be a maximal parabolic subdiagram. Each connected component Γ_s is an extended Dynkin diagram of $\widetilde{A}\widetilde{D}\widetilde{E}$ type. It is well known that there exists a unique primitive vector $n(\Gamma_s) = \sum n_i r_i$ with $r_i \in \Gamma_s$, $n_i \in \mathbb{N}$, such that $n(\Gamma_s) r_j = 0$ for all $r_j \in \Gamma_s$. This is a nonzero vector in the lattice N_2 that generates the corresponding type II ray. Thus, if Γ has ≥ 2 components then we get relations $n(\Gamma_s) = n(\Gamma_{s'})$. We call these *Type II relations*, and we record them below (plus the relations obtained from these by an S_3-symmetry). The weights are also shown in Figure 2.

$$\begin{aligned}
2\widetilde{E}_8\widetilde{A}_1:\quad & 6\vec a_0 + 4\vec a_1 + 2\vec a_2 + 3\vec b_0 + 5\vec a_{17} + 4\vec a_{16} + 3\vec a_{15} + 2\vec a_{14} + \vec a_{13} = \\
& 6\vec a_6 + 4\vec a_5 + 2\vec a_4 + 3\vec b_6 + 5\vec a_7 + 4\vec a_8 + 3\vec a_9 + 2\vec a_{10} + \vec a_{11} = \\
& \vec b_{12} + \vec d_3 \\
\widetilde{D}_{10}\widetilde{E}_7:\quad & \vec a_{17} + \vec b_0 + 2(\vec a_0 + \cdots \vec a_6) + \vec b_6 + \vec a_7 = \\
& 4\vec a_{12} + 2\vec b_{12} + 3\vec a_{11} + 3\vec a_{13} + 2\vec a_{10} + 2\vec a_{14} + \vec a_9 + \vec a_{15} \\
\widetilde{D}_{16}\widetilde{A}_1:\quad & \vec a_1 + \vec b_0 + 2(\vec a_0 + \cdots + \vec a_6) + \vec b_6 + \vec a_5 = \\
& \vec a_3 + \vec d_3
\end{aligned}$$

Theorem 2.5. *The following holds:*

1. *(1) The lattice N_2 has rank 19 and determinant 2.*
2. *(2) The sublattice generated by the 19 vectors $\vec a_i$, $\vec b_0$ has index 3 in N_2.*
3. *(3) N_2 is generated by the 20 vectors $\vec a_i, \vec b_0, \vec b_6$ with a unique $2\widetilde{E}_8$-relation $n(\widetilde{E}_8^{(1)}) - n(\widetilde{E}_8^{(2)}) = 0$ between them.*
4. *(4) N_2 is also generated by the 21 vectors $\vec a_i, \vec b_i$, and the relations between them are generated by the three $2\widetilde{E}_8$-relations; the sum of these relations is zero.*
5. *(5) The dual lattice $M_2 = N_2^*$ is generated by N_2 and one of vectors $\vec d'_i = \frac{1}{2}\vec d_i$. In particular, M_2 is generated by $\vec a_i, \vec b_0, \vec b_6, \vec d'_3$ with two relations coming from $2\widetilde{E}_8$ and $\widetilde{D}_{16}\widetilde{A}_1$.*

Proof. (1) is by definition, (2,5) are direct computations, and (3,4) are easy. □

Theorem 2.6. *The lattice M_2 and the cone $\check\sigma^{\rm refl}$ can be embedded into a lattice $\mathbb{Z}^{21}$ with the basis of vectors $\vec s_i$ $(0 \le i < 18)$ and $\vec t_i$ $(i = 0, 6, 12)$ by setting*

$$\vec a_i = \vec t_{i-1} + \vec t_{i+1} - 2\vec t_i \ (i \ne 0, 6, 12), \quad \vec a_i = \vec t_{i-1} + \vec t_{i+1} + \vec s_i - 2\vec t_i \ (i = 0, 6, 12),$$
$$\vec b_i = \vec t_i - 2\vec s_i \ (i = 0, 6, 12), \qquad \vec d'_i = \vec t_i + \vec s_{i+9} \ (i = 3, 9, 15),$$

where the indices i are considered modulo 18. Moreover, M_2 is the kernel of a homomorphism $\mathbb{Z}^{21} \to \mathbb{Z}^2 \subset \mathbb{Z}^3$ defined by $\vec t_i \mapsto \vec u_i - \vec u_c$, $\vec s_i \mapsto \vec v_i - \vec u_c$, where the *vectors $\vec{u}_i, \vec{u}_c, \vec{v}_i$ are defined by*

$$\vec{u}_0 = (6,0,0), \quad \vec{u}_1 = (5,1,0), \quad \vec{u}_2 = (4,2,0), \quad \dots \quad \vec{u}_{17} = (5,0,1)$$
$$\vec{v}_0 = (4,1,1), \quad \vec{v}_6 = (1,4,1), \quad \vec{v}_{12} = (1,1,4), \quad \vec{u}_c = (2,2,2).$$

(cf. the intergral points in the 6-6-6 lattice triangle as in depicted in Figure 7).

Proof. We can embed M_2 into $\mathbb{Z}^{24}$ and the cone $\check{\sigma}$ into $\mathbb{R}^{24}$ by evaluating for each of the 24 generators $\vec{a}_i$, $\vec{b}_i$, $\vec{d}_i'$ of M_2 their dot products with the 24 generators $\vec{a}_i$, $\vec{b}_i$, $\vec{d}_i$ of N_2. If we denote the coordinates on $\mathbb{Z}^{24}$ by $\vec{a}_i^* = \vec{t}_i$, $\vec{b}_i^* = \vec{s}_i$, $\vec{d}_i^* = \vec{r}_i$ then we will get the above expressions with some additional $\vec{r}_i$. We then project to $\mathbb{Z}^{24}/\langle \vec{r}_i \rangle = \mathbb{Z}^{21}$, and the sublattice M_2 projects isomorphically.

It is immediate to check that all $\vec{a}_i, \vec{b}_i, \vec{d}_i'$ are in the kernel of $\mathbb{Z}^{21} \to \mathbb{Z}^2$. □

3. Moduli of stable pairs

3.1. **General theory.** We briefly recall the main definitions and results concerning complete moduli spaces of stable pairs, referring for details to [Kol13, Kol15, Ale06, Ale15].

Definition 3.1. A pair consisting of a normal variety X and an $\mathbb{R}$-Weil divisor $B = \sum b_i B_i$, with $0 \le b_i \le 1$ and effective $\mathbb{Z}$-Weil divisors B_i, has log canonical (lc) singularities if $K_X + B$ is $\mathbb{R}$-Cartier and for any resolution of singularities $f\colon Y \to X$ in the natural formula

$$K_Y = \pi^*(K_X + B) + \sum a_E E,$$

one has $a_E \ge -1$, with the sum going over all irreducible divisors E on Y, whether they are f-exceptional or not. In particular, for any irreducible nonexceptional divisor E on X one has $\sum b_i \operatorname{mult}_E B_i \le 1$.

Definition 3.2. A pair consisting of a reduced, but possibly reducible, variety X and an $\mathbb{R}$-Weil divisor $B = \sum b_i B_i$, with $0 \le b_i \le 1$ and effective $\mathbb{Z}$-Weil divisors B_i, has semi log canonical (slc) singularities if

(1) X satisfies Serre's S_2 condition,
(2) X has at worst double crossing singularities in codimension 1,
(3) no component of B_i contains any component of the double locus, and,
(4) denoting by $\nu\colon X^\nu \to X$ the normalization of X, the pair $(X^\nu, \nu^{-1}B +$ (double locus)) is log canonical.

Definition 3.3. A pair (X, B) of a connected reduced, but possibly reducible, projective variety X and an $\mathbb{R}$-divisor $B = \sum b_i B_i$ is *stable* if

(1) (X, B) has slc singularities,
(2) $K_X + B$ is ample.

Definition 3.4. Fix real numbers C, $0 < \epsilon < 1$, and a positive integer $N \in \mathbb{N}$. Define the moduli functor $\underline{M}_{\epsilon,C,N}$ from Schemes to Sets by setting $\underline{M}_{\epsilon,C,N}(\mathcal{S})$ to be the set of of flat families $(\mathcal{X}, \mathcal{B} = \epsilon\mathcal{B}_1) \to S$ up to isomorphisms over S, satisfying:

(1) $\mathcal{B}_1 \subset \mathcal{X}$ is a relative divisor, flat over $\mathcal{S}$.
(2) Every geometric fiber is a stable pair $(X, B = \epsilon B_1)$ with $(K_X + B)^2 = C$.
(3) Denoting by $j\colon \mathcal{U} \to \mathcal{X}$ the open subset where the sheaves $\omega_{\mathcal{X}/\mathcal{S}}$ and $\mathcal{O}_{\mathcal{X}}(\mathcal{B}_1)$ are invertible, the complement $\mathcal{X} \setminus \mathcal{U}$ has codimension ≥ 2 on each fiber.
(4) $j_*\mathcal{O}_{\mathcal{U}}(N\mathcal{B}_1)$ is invertible and relatively ample.

(5) $j_*\omega_{\mathcal{U}/\mathcal{S}}^{\otimes N}$ is invertible and relatively trivial.

Theorem 3.5. *For a sufficiently small $0 < \epsilon \ll 1$, and fixed C, there exists an N such that the functor $\underline{M}_{\epsilon,C,N}$ is proper and is coarsely represented by a proper algebraic space M over $\mathbb{C}$.*

Moreover, Kollár's theorem [Kol90] and Fujino's generalization to pairs [Fuj12] imply that M is a projective scheme over $\mathbb{C}$. Note: taking a small coefficient ϵ avoids some thorny problems when some one-parameter limits of divisors B_1 acquire embedded points.

3.2. **Application to K3 surfaces of degree 2.** As an application of the general theory, consider the moduli functor F_2 as in Section 2.1, parameterizing pairs (S, L), where S is a K3 surface with ADE (Du Val) singularities, and L is an ample invertible sheaf with $L^2 = 2$. Let $\widehat{S}$ the minimal resolution of singularities of S, and $\widehat{L}$ be the pullback of L, a big and nef invertible sheaf on $\widehat{S}$. Then one of the following holds:

(1) (Hyperelliptic case) The linear system $|\widehat{L}|$ is base point free. Then $|\widehat{L}|$ defines a generically 2-to-1 map $\widehat{S} \to \mathbb{P}^2$ which descends to a finite morphism $\pi\colon S \to \mathbb{P}^2$ of degree 2.
(2) (Unigonal case) $|\widehat{L}|$ has a base curve, a (-2)-curve, and the moving part is "composed of a pencil". Then $\widehat{S}$ comes with a generically 2-to-1 map to the Hirzebruch surface $\mathbb{F}_4$ which descends to a finite morphism $\pi\colon S \to \mathbb{F}_4^0$ of degree 2, where the surface $\mathbb{F}_4^0$ is the cone over a rational normal curve of degree 4; it is obtained from $\mathbb{F}_4$ by contracting the (-4)-section.

Either way, we get a double cover $\pi\colon S \to X$, where X is $\mathbb{P}^2$ of $\mathbb{F}_4^0$. Let D be the ramification curve of π. Then $D \in |3L|$ and $D^2 = 18$. For $0 < \epsilon \ll 1$, the pair $(S, \epsilon D)$ is klt (implying it is slc), $K_S \sim 0$, and $(K_S + \epsilon D)^2 = 18\epsilon^2$. Thus, the general theory (3.5) provides a functorial compactification.

We will be interested in the main irreducible component of this compactificaiton, which we will denote by $\overline{F}_2$. (Thus, we are only interested in the stable surfaces which can be obtained as one-parameter degenerations of K3 surfaces).

Let $\pi\colon S \to X$ be a 2-to-1 Galois cover of surfaces that have at worst double crossing singularities, and let $B = \pi(D)$ denote the branch divisor on the base X. Let S^0, X^0 denote the compatible complements of the "bad loci", where X, S are not Gorenstein and/or D, B are not Cartier. Then the Riemann-Hurwitz theorem implies that

$$2K_{S^0} \sim \pi_0^*(2K_{X^0} + B^0), \qquad K_{S^0} + 2\epsilon D^0 \sim_{\mathbb{Q}} \pi_0^* \left(K_{X^0} + (1/2 + \epsilon)\, B^0\right)$$

Lemma 3.6. *The pair $(S, 2\epsilon D)$ is stable $\iff$ the pair $\big(X, (\frac{1}{2} + \epsilon)B\big)$ is stable.*

Proof. See e.g. [AP12, Lemma 2.3]. □

Corollary 3.7. *The functorial compactification $\overline{F}_2$ of the K3 surfaces of degree 2 coincides with the functorial compactification of log canonical pairs $\big(X, (\frac{1}{2} + \epsilon)B\big)$ with $X = \mathbb{P}^2$ or $\mathbb{F}_4^0$, and $B \in |-2K_X|$.*

Moreover, since the pairs $(\mathbb{F}_4^0, B)$ of this kind are one-parameter limits of planar pairs, this is the same as the functorial compactification of log canonical pairs $\big(\mathbb{P}^2, (\frac{1}{2} + \epsilon)B\big)$ where B is a planar sextic.

3.3. **Hacking's compactification $\overline{M}(\mathbb{P}^2, d)$ for planar pairs.** The compactification of of the moduli space of log canonical pairs $(\mathbb{P}^2, (\frac{1}{2}+\epsilon)B)$ with B a sextic curve is a special case of the moduli space of *planar stable pairs of degree* d from [Hac01, Hac04]:

Definition 3.8. A *planar stable pair of degree* d is a pair (X, B) such that

- the pair $(X, (\frac{3}{d}+\epsilon)B)$ is stable, in the sense of Definition 3.3,
- the divisor $dK_X + 3B$ is linearly equivalent to zero, and
- there is a deformation $(\mathcal{X}, \mathcal{B})/S$ of (X, B) over the germ of a curve S, such that the general fiber X_t of $\mathcal{X}/S$ is isomorphic to $\mathbb{P}^2$ and the divisors $K_{\mathcal{X}}$ and $\mathcal{B}$ are $\mathbb{Q}$-Cartier.

It follows from the results above that the moduli space $\overline{M}(\mathbb{P}^2, d)$ of stable planar pairs of degree d is compact and agrees with the main irreducible component of the compactification of log canonical pairs $(\mathbb{P}^2, (\frac{1}{2}+\epsilon)B)$ when $d = 6$.

The moduli spaces $\overline{M}(\mathbb{P}^2, d)$ have been studied in detail by Hacking [Hac01, Hac04]. In particular, he has obtained partial classifications of the stable pairs that can appear. For later reference, we will briefly summarize his results here.

The first result that we will need is a description of the possible irreducible components of stable pairs of degree d. Let (X, B) denote such a stable pair and let Y be an irreducible component of the normalization of X. Denote the inverse image of the double curve in X by C. Then Definition 3.3 implies that $-(K_Y + C)$ is ample and the pair (Y, C) is log canonical.

Theorem 3.9. [Hac04, Theorem 5.3] *Let Y be a surface and let C be an effective divisor on Y such that the pair (Y, C) is log canonical and $-(K_Y + C)$ is ample. Then (Y, C) is one of the following types:*

(I) $C = 0$ *and Y has at most one strictly log canonical singularity;*
(II) $C \cong \mathbb{P}^1$ *and (Y, C) is log terminal;*
(III) $C \cong \mathbb{P}^1 \cup \mathbb{P}^1$*, where the components of C meet at a single node and the pair (Y, C) is log terminal away from the node;*
(IV) $C \cong \mathbb{P}^1$ *and (Y, C) has a singularity with the local form $(\frac{1}{r}(1, a), (xy = 0))/\mu_2$, where the μ_2-action is étale in codimension 1 and exchanges $(x = 0)$ and $(y = 0)$. Moreover, (Y, C) is log terminal away from this singularity.*

Using this, Hacking goes on to give a list of possible slc surfaces X that may underlie stable pairs (X, B) of degree d. To state his result, we first give some notational conventions and terminology.

Let X be an slc surface and let D denote the double curve on X. Let $X_1, \ldots, X_n$ be the irreducible components of X and let D_i be the restriction of D to X_i. Let $\nu\colon (X^\nu, D^\nu) \to (X, D)$ denote the normalization of X and let (X_i^ν, D_i^ν) denote its irreducible components.

The map $D^\nu \to D$ is $2:1$. Let $\Gamma \subset D$ be any component and write Γ^ν for its inverse image on X^ν. If Γ^ν is irreducible and is a double cover of Γ, we say that $\Gamma^\nu \subset X^\nu$ is *folded* to obtain $\Gamma \subset X$.

Theorem 3.10. [Hac04, Theorem 5.5] *Let X be an slc surface so that $-K_X$ is ample. Then X has one of the following types.*

(A) *X is normal, with one component of type (I).*
(B) *X has two components X_1, X_2 such that (X_i^ν, D_i^ν) is of type (II) for $i = 1, 2$.*

(B*) *X is irreducible and non-normal. The pair (X^ν, D^ν) is of type (II) and X is obtained by folding the curve D^ν.*

(C) *X has n components $X_1, \ldots, X_n$ such that (X_i^ν, D_i^ν) is of type (III) for each i. One component of D_i^ν is glued to a component of $D_{i+1 \bmod n}^\nu$ for each i so that the nodes of the curves D_i^ν coincide and the components X_i of X form an* **"umbrella"**.

(D) *X has n components $X_1, \ldots, X_n$ such that (X_i^ν, D_i^ν) is of type (III) for each $2 \leq i \leq n-1$. Either (X_1^ν, D_1^ν) is of type (IV), or (X_1^ν, D_1^ν) is of type (III) and (X_1, D_1) is obtained by folding one component of D_1^ν; similarly for (X_n, D_n). The components $(X_1, D_1), \ldots, (X_n, D_n)$ are glued sequentially so that the nodes of the curves D_i^ν and any strictly log canonical singularities in (X_1, D_1) and (X_n, D_n) coincide, and the components X_i of X form a* **"fan"**.

Furthermore, Hacking later shows [Hac04, Theorem 6.5] that surfaces of type (B*) do not admit smoothings to $\mathbb{P}^2$, so cannot possibly underlie stable pairs of degree d. Hence all stable pairs of degree d must fall into one of the classes (A), (B), (C) or (D).

In the case of interest to us, $d = 6$, Hacking is able to go even further. In [Hac01, Section 14.3.1], he classifies stable pairs of degree 6 of types (A) and (B) (this classification is reproduced in Table 1). Furthermore, in [Hac01, Section 14.3.2], he gives an upper bound list of the irreducible components (Y, C) of types (III) and (IV) that may occur in stable pairs of types (C) and (D), and gives constraints restricting how such components may be glued to give stable pairs of degree 6.

Surface	Double Curve
$\mathbb{P}^2$	
$\mathbb{F}_4^0 = \mathbb{P}(1,1,4)$	
Elliptic cone, degree 9	
$\mathbb{P}^2 \cup \mathbb{F}_1$	line $\cup$ (-1)-section
$\mathbb{P}^2 \cup \mathbb{F}_4$	conic $\cup$ (-4)-section
$\mathbb{P}(1,1,4) \cup \mathbb{F}_4$	quartic $\cup$ (-4)-section
$\mathbb{P}(1,1,2) \cup \mathbb{P}(1,1,2)$	line $\cup$ line

TABLE 1. Classification of stable pairs of degree 6 of types (A) and (B), as given by Hacking [Hac01, Section 14.3.1]

Remark 3.11. In order to complete the classification of stable pairs of degree 6, it remains to show which of these stable pairs of types (C) and (D) admit smoothings to $\mathbb{P}^2$. This will be addressed in the next section, where we will give a complete classification of the stable pairs of the "umbrella" type (C) which admit smoothings to $\big(\mathbb{P}^2, (\frac{1}{2} + \epsilon)B\big)$ and show that stable pairs of the "fan" type (D) do not occur.

For later reference, we give Hacking's list of possible components (Y, C) of type (III) (which form the irreducible components of stable pairs of degree 6 of type (C)) in Table 2, along with their Picard numbers ρ.

The notation used in this table is as follows: $\mathrm{Bl}_{(m,n)}$ denotes the weighted blow-up of a smooth point on a surface with weights (m, n) with respect to some local analytic coordinates; unless otherwise stated it will be assumed that the point

Case	Surface	Parameter	Double Curve	ρ
1	$\mathbb{F}_n^0 = \mathbb{P}(1,1,n)$	$1 \le n \le 9$	$H_1 + H_2$	1
2	$\mathbb{P}(1,2,2n-1)$	$1 \le n \le 9$	$H + (2H)$	1
3	$\mathbb{P}(1,1,2(n-1))/\mu_2$	$2 \le n \le 10$	$H_1/\mu_2 + H_2/\mu_2$	1
4	$\mathbb{F}_n$	$0 \le n \le 7$	$F + S$	2
5	$\mathbb{F}_{n-\frac{1}{2}}$	$1 \le n \le 7$	$\frac{1}{2}F + S$	2
6	$\mathrm{Bl}_{(1,2)}\mathbb{P}(1,1,n)$	$2 \le n \le 9$	$H_1' + H_2$	2
7	$\mathrm{Bl}_{(1,2)}\mathbb{P}(1,2,2n-1)$	$2 \le n \le 10$	$H + (2H)'$	2
8	$\mathrm{Bl}_{(1,1)}\mathbb{P}(1,1,2)$	$n = 2$	$H' + E$	2
9	$\mathrm{Bl}_{(1,2)}\mathbb{P}(1,1,2)$	$n = 2$	$H' + E$, $\mathrm{wt}(H) = 2$	2
10	$\mathrm{Bl}_{(2,3)}\mathbb{P}(1,1,2)$	$n = 2$	$H' + E$	2
11	$\mathrm{Bl}_{(1,2)}\mathbb{F}_{n-\frac{1}{2}}$	$n = 1$	$\frac{1}{2}F + (S + \frac{1}{2}F)'$, $\mathrm{wt}(S) = 1$	3
	$\mathrm{Bl}_{(1,2)}\mathbb{F}_{n-\frac{3}{2}}$	$2 \le n \le 9$	$\frac{1}{2}F + S'$	3
12	$\mathrm{Bl}_{(1,2)}\mathbb{F}_{n-1}$	$2 \le n \le 8$	$F + S'$, $\mathrm{wt}(S) = 1$	3
13	$\mathrm{Bl}_{(1,2)}\mathbb{F}_n$	$2 \le n \le 7$	$F' + S$	3
14	$\mathrm{Bl}^2_{(1,2)}\mathbb{P}(1,1,n)$	$5 \le n \le 10$	$H_1' + H_2'$	3

Table 2. Candidates for type (III) components of stable pairs of degree 6, as given by Hacking [Hac01, Section 14.3.2]

and choice of analytic coordinates is general. H denotes a general hyperplane section and E will always denote an exceptional curve. Parentheses denote a general member of a linear system (so $(2H)$ denotes a general member of the linear system $|2H|$) and strict transforms of divisors are denoted by primes (so the strict transform of H under a blow-up will be denoted H'; in this case it is always assumed that H passes through the centre of the blow-up).

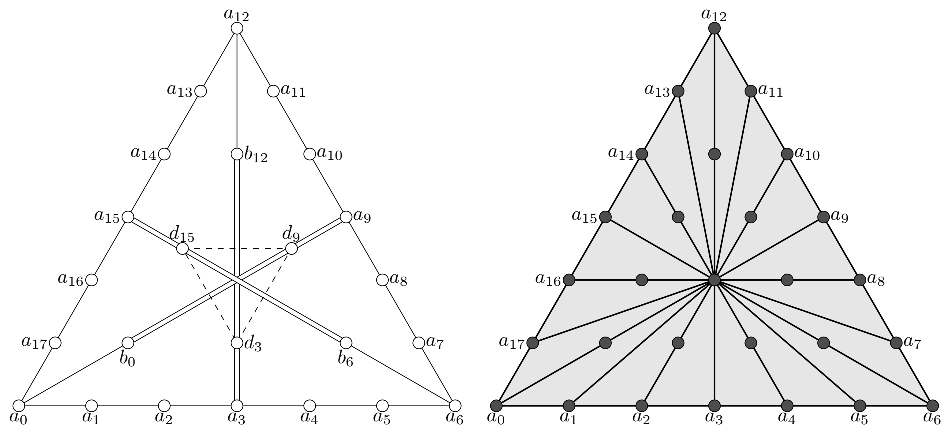
$\mathbb{F}_{n-\frac{1}{2}}$, for $n \ge 1$, denotes the surface obtained from $\mathbb{F}_n$ in the following way: first perform a sequence of two blowups away from the negative section, to obtain a degenerate fiber which is a chain of curves with self-intersections -2, -1, -2, then contract the two (-2)-curves. Thus $\mathbb{F}_{n-\frac{1}{2}}$ has one double fiber with two A_1 singularities on it and a negative section with square $-(n-\frac{1}{2})$. Denote the negative section and fiber in $\mathbb{F}_n$ or $\mathbb{F}_{n-\frac{1}{2}}$ by S and F respectively, and write $\frac{1}{2}F$ for the double fiber with its reduced structure.

4. Limits of sextic pairs and the Vinberg diagram Γ_{Vin}

In this section, we classify all stable limits of sextic pairs $\big(\mathbb{P}^2, (\frac{1}{2}+\epsilon)B\big)$, i.e. the pairs appearing on the boundary of the moduli compactification $\overline{F}_2$.

4.1. **Statement of the result.** Consider Figure 3b which represents a toric degeneration of a $\mathbb{P}^2$ together with a sextic curve B. The points correspond to monomials of degree 6: $a_0 \mapsto x^6$, $a_1 \mapsto x^5y$, etc., the center is $x^2y^2z^2$.

The surface $X = \cup_{i=1}^{18}\mathbb{F}_2^0$ represented by this figure is a union of 18 quadratic cones $\mathbb{F}_2^0$ (Hirzebruch surface $\mathbb{F}_2$ with the (-2)-section contracted). The vertices of the cones are on the "outside" boundary at the "odd" vertices $P_1, P_3, \dots, P_{17}$, and the surfaces are glued in an alternating order along $\mathbb{P}^1$s. We divide these $\mathbb{P}^1$'s into "long" (corresponding to even vertices) and "short" (corresponding to odd vertices) sides; they are respectively infinite sections s_∞ with $s_\infty^2 = 2$ (a conic section) and



(A) Vinberg diagram (B) Maximal degeneration

FIGURE 3. Vinberg diagram vs maximal degeneration

lines f through the vertex (the image of a fiber on $\mathbb{F}_2$) with $f^2 = \frac{1}{2}$. The restriction $B_i = B|_{X_i}$ is another conic section.

The corresponding degeneration of K3 surfaces is the double cover $S = \cup_{i=1}^{18} S_i$ ramified over the curve B and the 9 points $P_1, P_3, \dots, P_{17}$, which are the vertices of the cones lying on the short sides. Each irreducible component S_i is isomorphic to a $\mathbb{P}^2$. These 18 $\mathbb{P}^2$s are glued in an alternating order along “long” and “short” sides, which are conics and lines respectively.

Theorem 4.1. *There exists a unique maximal stable degeneration of the sextic pairs $\big(\mathbb{P}^2, (\frac{1}{2}+\epsilon)B\big)$ and of the correponding K3 pairs $(S, 2\epsilon D)$. All other stable degenerations are obtained from it by partial smoothings according to the Rules 4.2. The degeneration types of surfaces correspond to elliptic and maximal parabolic subdiagrams $\Gamma \subset \Gamma_{\rm Vin}$ modulo the equivalence relation defined in (4.6).*

Rules 4.2. Let $\Gamma \subset \Gamma_{\rm Vin}$ be an elliptic or maximal parabolic subdiagram, as in the description 2.2 of the cones in the fan $\tau^{\rm refl}$.

(1) All irreducible components S_i are toric surfaces with two exceptions: the one specified in rule (5) and when $\#(i \mid a_i \in \Gamma) \geq 17$. In the latter case, S is irreducible. If $\Gamma = \widetilde{A}_{17}$ then S is a cone over an elliptic curve. Otherwise, S is obtained from a toric surface by gluing two “opposite” sides.

(2) Each toric surface corresponds to a polytope P that is a union of adjacent triangles in Figures 3b, 4. The two triangles near a corner of the 6-6-6 (or 3-3-12) triangle can be glued in two ways: by creating, resp. not creating a corner; cf. the two triangles adjacent to the a_{12} edge in Figure 3b, resp. 4.

(3) If $a_i \in \Gamma$ then we smooth the curve corresponding to the edge from the center to the outside vertex. This edge between triangles is removed to obtain a bigger polytope.

(4) For $i \in \{0, 6, 12\}$, suppose $a_i \in \Gamma$, so that the a_i-edge is smoothed out. Then if $b_i \in \Gamma$ then the corresponding polytope has a corner at a_i, as in Figure 3b. But if $b_i \notin \Gamma$ then the polytope has a straight line at a_i, as for $i = 12$ in Figure 4.

(5) For $i \in \{3,9,15\}$, suppose $a_i \in \Gamma$, so that the a_i-edge is smoothed out, and that $d_i \in \Gamma$. Then the connected component of Γ containing a_i, d_i must equal $\widetilde{A}_1$. The corresponding irreducible component S_k is isomorphic to either $\mathbb{P}^2$ (non-unigonal case) or $\mathbb{F}_4^0$ (unigonal case), and the intersection $C = S_k \cap (\cup_{j\neq k} S_k)$ is either a conic on $\mathbb{P}^2$ or a section s_∞ on $\mathbb{F}_4^0$ with $s_\infty^2 = 4$. The partial smoothing in this case is of two lines $\mathbb{P}^1 \cup \mathbb{P}^1$ for the edges $a_{i\pm1}$ to the irreducible curve C.

Note that $\mathbb{F}_4^0$ is toric. Although $\mathbb{P}^2$ is also a toric surface, the conic C in this case is not a torus-invariant divisor.

(6) Connected components of the diagram Γ not supported entirely on the set $\{b_0, b_6, b_{12}, d_3, d_9, d_{15}\}$ correspond to the irreducible components X_i of X which are bigger than a single $\mathbb{F}_2^0$. (For notational consistency, we formally assign an $\mathbb{F}_2^0$ component to a diagram $A_0 = \emptyset \subset \Gamma$.)

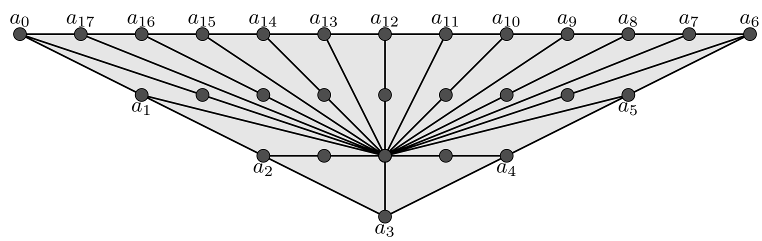


FIGURE 4. Unigonal maximal degeneration

Remark 4.3. All stable surfaces claimed in Thereom 4.1 are of the "umbrella" type from (3.10). Thus, we claim that the surfaces of the "fan" type do not occur.

Remark 4.4. Suppose that $a_i \notin \Gamma$, and $i \in \{0,6,12,3,9,15\}$. Then the corresponding a_i-edge has not been smoothed out and the degenerate surface is the same whether the corresponding b_i or d_i is in the diagram Γ or not. Thus, the b_i and d_i vertices in Γ play a role only if they are attached to an a_i vertex in Γ.

Remark 4.5. Although the degenerate surfaces in Figures 3b and 4 may look different, they are isomorphic. Both are simply 18 copies of $\mathbb{F}_2^0$ glued together along $\mathbb{P}^1$s to form a seminormal surface. This seminormal surface is unique: how one draws it is not important.

Definition 4.6 (Equivalence relation)**.** We define the equivalence relation on the set of elliptic and maximal parabolic subdiagrams $\Gamma \subset \Gamma_{\mathrm{Vin}}$ as follows:

1. if Γ, Γ' are disconnected from each other and Γ' is supported on the set $\{b_0, b_6, b_{12}, d_3, d_9, d_{15}\} \implies \Gamma \sim \Gamma \sqcup \Gamma'$.
2. Γ_1 and Γ_2 differ by an S_3-action $\implies \Gamma_1 \sim \Gamma_2$.
3. Γ_1 and Γ_2 are both contained in the $\widetilde{A}_{17}$ subdiagram formed by the 18 outer a_i vertices and differ by a D_9-action $\implies \Gamma_1 \sim \Gamma_2$.

Example 4.7. In Figure 5 we illustrate how this correspondence works for the maximal parabolic subdiagrams of Figure 2.

For $\Gamma = 2\widetilde{E}_8\widetilde{A}_1$, the two polytopes are obtained by cutting the 6-6-6 triangle of Figure 3b by a vertical line. All edges away from this line are smoothed, and the corresponding stable surface is $X = \mathbb{F}_2^0 \cup \mathbb{F}_2^0$. For $\Gamma = \widetilde{D}_{10}\widetilde{E}_7$, the 6-6-6 triangle is cut by a horizontal line, and $X = \mathbb{P}^2 \cup \mathbb{F}_1$. For $\Gamma = \widetilde{D}_{16}\widetilde{A}_1$, the 3-3-12 triangle of Figure 4 is cut by a horizontal line giving $X = \mathbb{F}_4 \cup \mathbb{P}^2$ with the boundary between the two components being a conic. Finally, for $\Gamma = \widetilde{A}_{17}$, all 18 edges are smoothed out and there are no corners; the corresponding surface X is a cone over an elliptic curve. Noting that $\mathbb{P}(1,1,4) \cup \mathbb{F}_4$ is a degeneration of $\mathbb{F}_4 \cup \mathbb{P}^2$, we thus obtain all of the expected Type II degenerations from Table 1.

Remark 4.8. Not all cases of candidate type (III) components from Table 2 appear as components of stable degenerations in Theorem 4.1. Table 3 shows for which parameter n the candidate components do appear, and gives the corresponding ellliptic subdiagrams $\Gamma \subset \Gamma_{\mathrm{Vin}}$. For clarity, we will henceforth refer to irreducible components of degenerate fibers by the names given in the "Name" column of this table, which makes the correspondence with elliptic subdiagrams explicit.

Note here that, up to the action of D_9, there are two different ways to embed an A_m diagram, with m odd, into the $\tilde{A}_{17}$ diagram formed from the outer vertices. We thus say that an A_m diagram, with m odd, that is supported on $\tilde{A}_{17}$ is of *corner type* if it is equivalent under the action of D_9 to one that starts in a corner of Γ_{Vin}, and of *edge type* otherwise. Using the terminology above, a diagram A_m with m odd is of corner type if the corresponding component has two "short" sides, and edge type if it has two "long" sides.

Finally, we recall that the components $\mathbb{F}_2^0$ making up the maximal degeneration (Figure 3b) formally correspond to $A_0 = \emptyset$ diagrams. The maximal degeneration thus corresponds to the unique empty subdiagram of Γ_{Vin}.

Case	Parameter	Elliptic subdiagram	Name	ρ
1	$1 \le n \le 9$	$A_{2n-1} \subset \tilde{A}_{17}$ of edge type	A^e_{2n-1}	1
2	$1 \le n \le 9$	$A_{2n-2} \subset \tilde{A}_{17}$	A_{2n-2}	1
3	$2 \le n \le 10$	$A_{2n-3} \subset \tilde{A}_{17}$ of corner type	A^c_{2n-3}	1
4	$0 \le n \le 7$	D_{2n+4} contains 1 interior vertex	D_{2n+4}	2
5	$1 \le n \le 7$	D_{2n+3} contains 1 interior vertex	D_{2n+3}	2
6	$2 \le n \le 9$	A_{2n-1} contains 1 interior vertex	A'_{2n-1}	2
7	$2 \le n \le 10$	A_{2n-2} contains 1 interior vertex	A'_{2n-2}	2
8	$n = 2$	E_7	E_7	2
9	$n = 2$	E_8	E_8	2
10	$n = 2$	E_6	E_6	2
11	impossible			3
12	impossible			3
13	$n \in \{3, 6\}$	D_{2n+4} contains 2 interior vertices	D'_{2n+4}	3
14	$n \in \{5, 8\}$	A_{2n-1} contains 2 interior vertices	A''_{2n-1}	3

TABLE 3. Correspondence between cases from Table 2 and elliptic subdiagrams of Γ_{Vin}

4.2. **Two toric families.** We begin the proof of Theorem 4.1 by constructing two 16-dimensional "toric" families in which many (but not all) of the stable surfaces

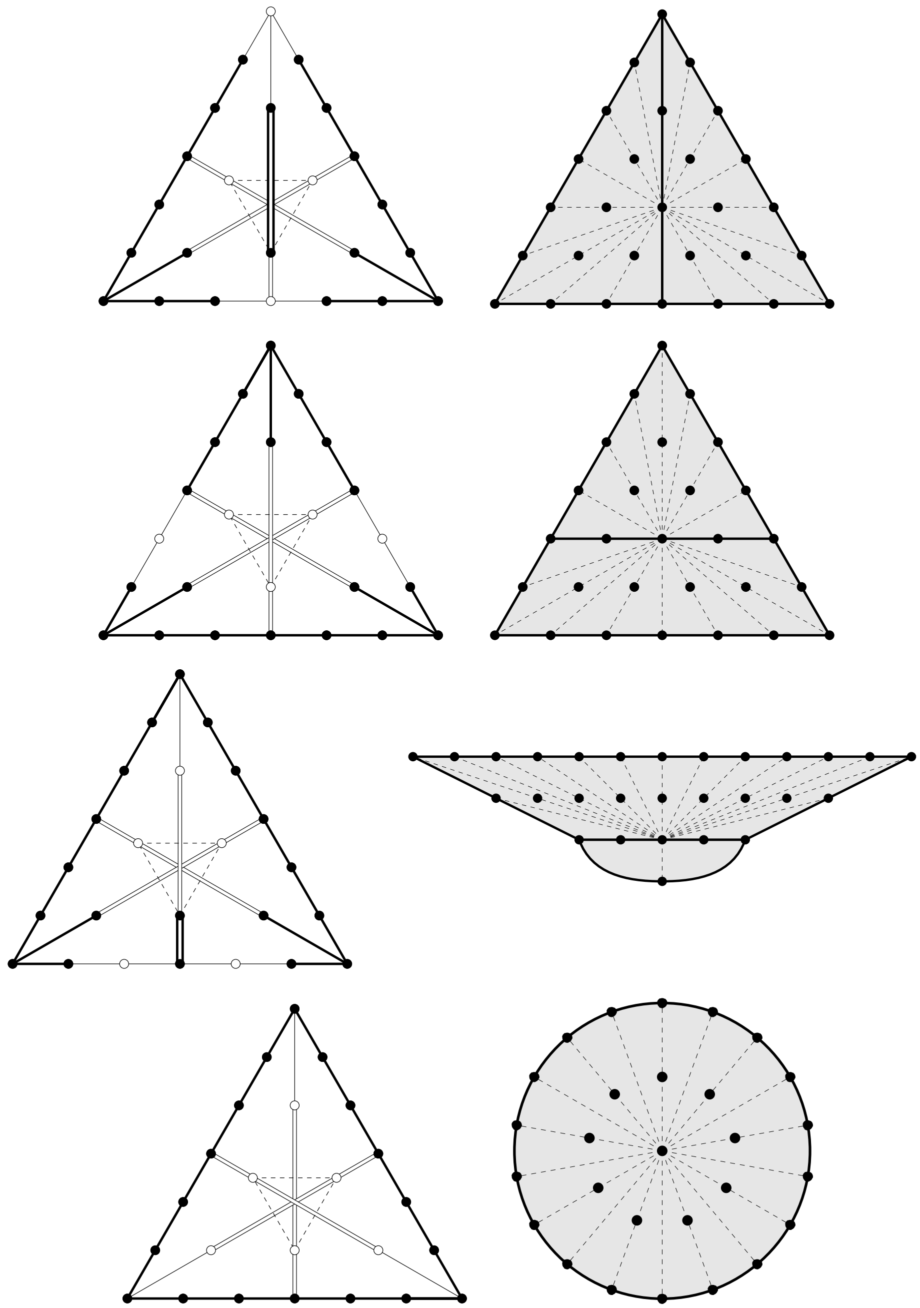


Figure 5. Maximal parabolics in Γ_{Vin} and Type II degenerations

already occur. The construction is standard and is based on the theory of secondary polytopes, see [GKZ94]. Let A be a finite subset in $\mathbb{Z}^r$ and let Q be the polytope which is the convex hull of A. In [KSZ92], the authors constructed a family of possibly non-normal toric varieties $X'_Q \subset \mathbb{P}^{|A|-1}$ and their degenerations over a possibly non-normal projective toric variety corresponding to a secondary polytope $\Sigma(Q, A)$. This construction works as follows: the polytope Q defines a polarized toric variety (X_Q, L_Q) and A defines a basepoint free linear system $F(c_m) = \langle c_m x^m;\ m \in A\rangle \subset H^0(X_Q, L_Q)$. An irreducible fiber $X'_Q(c_m)$ of the above family is then the image of the finite morphism $\phi_{F(c_m)} : X_Q \to \mathbb{P}^{|A|-1}$; it comes with multiplicity $\deg \phi_{F(c_m)}$.

A version of this construction that is needed for our purposes was given in [Ale02]. The subvariety $X'_Q(c_m)$ is replaced by a pair (X_Q, B), $B = Z(\sum c_m x^m)$ consisting of a normal projective toric variety X_Q and a divisor B on it which does not contain any torus orbits. One obtains a flat family $(\mathcal{X}, \mathcal{B}) \to \overline{\mathcal{M}}(Q, A)$ over a stack with a coarse moduli space $\overline{M}(Q, A)$ whose normalization is the normal toric variety correponding to the secondary polytope $\Sigma(Q, A)$. (This moduli space generally also has other irreducible components; we ignore those for simplicity.)

Degenerations of the toric variety X'_Q, which in [KSZ92, GKZ94] are considered simply as cycles, are replaced by reduced seminormal varieties X with torus action along with Cartier divisors B not containing any torus orbits. It follows that the pair $(X, \Delta + \epsilon B)$ is stable in the sense of (3.3), where Δ is the "outside" torus boundary, described in toric terms by the boundary of Q.

Here is how we apply this construction in our situation. Consider the set A of 19 points, 18 outer points plus the center, of Figure 3b. The toric variety for the secondary polytope is covered by affine charts $U_{\mathcal{T}}$, one for each triangulation $\mathcal{T}$ of Q with vertices in a subset $A' \subset A$. We pick the chart for the triangulation $\mathcal{T}_0$ shown in Figure 3b.

Lemma 4.9. *The affine toric variety $U_{\mathcal{T}_0}$ is a 16-dimensional toric subvariety of the toric variety U^{refl} of Definition 2.3.*

Proof. By the general theory of [GKZ94], to every triangulation $\mathcal{T} = \{\delta_i\}$ of the set A one associates a vector $v_{\mathcal{T}}$ in the lattice $\mathbb{Z}^A$: $v_{\mathcal{T}}(a) = \sum_{a\in\delta_i} \operatorname{Vol}\delta_i$ if $a \in A'$, $v_{\mathcal{T}}(a) = 0$ if $a \notin A'$. These vectors span an affine translate of a lattice $M \simeq \mathbb{Z}^d$, where $d = |A| - \dim Q - 1$. The secondary polytope $\Sigma(Q, A)$ is the convex hull of $v_{\mathcal{T}}$'s.

Thus, the cone $\check{\sigma} \subset M \otimes \mathbb{R}$ corresponding to $U_{\mathcal{T}_0}$ is spanned by the vectors $v_{\mathcal{T}} - v_{\mathcal{T}_0}$ where $\mathcal{T}$ goes over the "neighboring" triangulations of A that differ from $\mathcal{T}_0$ by a flip. For the triangulation $\mathcal{T}_0$ there are 18 neighboring triangulations, one for each of the boundary points. The dimension of the lattice M is $d = 19-2-1 = 16$. Thus, we get 18 vectors $v_{\mathcal{T}} - v_{\mathcal{T}_0}$ in an 16-dimensional space generating the cone $\check{\sigma}$. We write these vectors explicitly in the next Section.

Using the linear relations of Section 2.3, one checks that $\check{\sigma}$ is the image of the cone $\check{\sigma}^{\mathrm{refl}}$ under the linear map $M_2 \twoheadrightarrow M = M_2/\langle \vec{b}_0, \vec{b}_6, \vec{b}_{12}\rangle$. For the dual lattices, we obtain $N \subset N_2$. Thus, $U_{\mathcal{T}_0}$ is a toric subvariety of U^{refl} which is invariant under the action of the 16-dimensional torus $N \otimes \mathbb{C}^* = \operatorname{Hom}(M, \mathbb{C}^*)$. □

Lemma 4.10. *Over an open neighborhood of the origin of $U_{\mathcal{T}_0}$ (the closed torus orbit), there exists a flat family of stable pairs $\big(X, (\frac{1}{2}+\epsilon)B\big)$ extending the family of lc sextic pairs $\big(\mathbb{P}^2, (\frac{1}{2}+\epsilon)B\big)$.*

Proof. As we explained above, by the general theory of [Ale02], over $U_{\mathcal{T}_0}$ there is a flat family of stable pairs $\big(X, \Delta + \epsilon B\big)$ extending the family of $\big(\mathbb{P}^2, \Delta + \epsilon B\big)$, where Δ is the outer toric boundary $xyz = 0$ and B is a family of sextics not passing through the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, i.e. whose equations have nonzero coefficients for the monomials x^6, y^6, z^6.

The central fiber $\big(X_0, \Delta_0 + \epsilon B_0\big)$ of this family is described by the triangulation $\mathcal{T}_0$, which is a union of 18 $\mathbb{F}_2^0$'s, and the divisor B, whose restriction to each irreducible component is a conic section s_∞ not passing through the vertex of the cone or the central point where the 18 components meet. Now we observe the following:

(1) $\big(X_0, (\frac12 + \epsilon)B_0\big)$ is also a stable pair.

(2) Let X be any partial smoothing of X_0 not isomorphic to $\mathbb{P}^2$, and let X_P be an irreducible component of X. It corresponds to a lattice polytope P in the plane which contains the central point c on its boundary. Now the fact that the outside boundary of P is at lattice distance 2 from c implies that $2\Delta \sim B$.

This implies that over an open neighborhood of 0 the pairs $\big(X, (\frac12 + \epsilon)B\big)$ are also stable. □

Remark 4.11. In the above theorem, we are simplifying slightly. Because of the nontrivial but finite automorphism group of the central fiber, to get a family of reduced varieties one has to make a finite ramified μ_2^{16}-base change $U'_{\mathcal{T}_0} \to U_{\mathcal{T}_0}$. Then there is a family of stable pairs over $U'_{\mathcal{T}_0}$ and over the stack $[U'_{\mathcal{T}_0} : \mu_2^{16}]$ with the coarse moduli space $U_{\mathcal{T}_0}$.

The same exact argument works for the family of stable toric varieties for the triangulation of the set of 19 points, the center plus the outer 18 points, of Figure 4. The general fiber in this family is a pair $(\mathbb{F}_4^0, B)$, $B \in |-2K_{\mathbb{F}_4^0}|$. Again, the central fiber is a stable pair for both $\Delta_0 + \epsilon B_0$ and for $(\frac12 + \epsilon)B_0$, and for all smoothings one has $\Delta \sim \frac12 B$. In this case, the 16-dimensional lattice M is a quotient of M_2 modulo the sublattice $\langle \vec{b}_0, \vec{b}_6, \vec{d}_3\rangle$.

Lemma 4.12. *Let $\cup P_i$ be a subdivision of the 6-6-6 triangle of Figure 3b or the 3-3-12 triangle of Figure 4 obtained by gluing some of the 18 triangles to form ≥ 2 convex polytopes P_i. Let (X_0, B_0) be the correponding stable surface, as in Theorem 4.1. Then it appears as a stable degeneration of some 1-parameter family of pairs $\big(\mathbb{P}^2, (\frac12 + \epsilon)B\big)$ or $\big(\mathbb{F}_4^0, (\frac12 + \epsilon)B_0\big)$.*

Proof. This holds by by the toric construction [KSZ92, Ale02]. We pick heights h_m for the 19 points in Figures 3b or 4 so that the projection of the lower convex envelope of the points (m, h_m) gives the decomposition $\cup P_i$. Then the 1-parameter limit of the family (X_t, B_t) with $B_t = Z(\sum t^{h_m} x^m)$ is a stable toric variety $\cup (X^{(i)}, L^{(i)})$, where $(X^{(i)}, L^{(i)})$ is the polarized toric variety corresponding to the lattice polytope P_i. □

Remark 4.13. We may easily describe the stable surfaces arising from this lemma in terms of the correspondence between degeneration types and elliptic/maximal parabolic subdiagrams $\Gamma \subset \Gamma_{\mathrm{Vin}}$ given in Theorem 4.1. Indeed, stable surfaces arising as degenerations of $(\mathbb{P}^2, (\frac12 + \epsilon)B)$ in this lemma correspond to those subdiagrams Γ that contain all three vertices b_i (with vertices labeled as in Figure 3a), whilst stable surfaces arising as degenerations of $\big(\mathbb{F}_4^0, (\frac12 + \epsilon)B_0\big)$ correspond to those subdiagrams Γ that contain b_0, b_6, and d_3 (up to equivalence (4.6)).

Thus, the only subdiagrams Γ missing from this construction are those which contain few of the interior vertices b_i, d_i. Moreover, by the equivalence relation (4.6), we may freely add or remove subdiagrams Γ' that are supported on these interior vertices, as long as Γ' and Γ are not connected. It follows that any subdiagrams missing from the above construction must contain "long" elliptic subdiagrams, that contain one or more of the vertices a_0, a_6, a_{12} without containing the corresponding b_0, b_6, b_{12}.

4.3. **Parametric equations for toric families.** For later generalizations, we need to recall the equations for the toric families in more detail. The most convenient equations are parametric.

Let us start with the toric family for the 6-6-6 triangle. We will begin by describing 1-parameter degenerations. The data for such a degeneration is a *system of heights* $\{T_c,\ T_i\,(0 \le i < 18\}$, one for each of the 1+18 points in Figure 3b. For convenience, we set $T_c = 0$.

For each of the 1+18 points in the plane, we have a corresponding vector. As in Theorem 2.6, let us denote them by $\vec{u}_c, \vec{u}_i$. Consider the convex hull of the points $(\vec{u}_c, T_c)$, $(\vec{u}_i, T_i)$ in $\mathbb{Z}^{2+1}$. The projection of the *lower convex hull* of these points down to $\mathbb{R}^2$ defines a subdivision of the 6-6-6 triangle into a face-fitting tiling $\cup Q_i$ into polytopes.

The 1-parameter families of stable pairs $(\mathbb{P}^2, \Delta + \epsilon B)$ whose limits lie in $U_{\mathcal{T}_0}$ correspond to the system of heights such that $\cup Q_i$ is either the triangulation $\mathcal{T}_0$ or its coarsening. The corresponding family can be explicitly described as $\operatorname{Proj} R \to \operatorname{Spec} \mathbb{C}[t]$, where R is the subring of the ring $\mathbb{C}[t][x,y,z]$, graded by $\deg x = \deg y = \deg z = 1$, which is generated by the monomials

$$u_c = x^2y^2z^2,\ u_0 = t^{T_0}x^6,\ u_1 = t^{T_1}x^5y,\ \dots,\ u_{17} = t^{T_{17}}x^5z.$$

More carefully, to get a reduced central fiber, one may have to make a ramified finite base change $t = t_1^m$ for some divisible m.

The conditions for the heights to define a tiling $\cup Q_i$ that is a coarsening of the triangulation $\mathcal{T}_0$ are

$$T_{i-1} + T_{i+1} - 2T_i \ge 0 \ (i \ne 0, 6, 12),$$
$$T_{i-1} + T_{i+1} - \tfrac{3}{2}T_i \ge 0 \iff T_{i-1} + T_{i+1} + S_i - 2T_i \ge 0,\ S_i = \tfrac{1}{2}T_i \ (i = 0, 6, 12).$$

Let $t_i = t^{T_i}$ and $s_i = t^{S_i}$. Denote $a_i = \frac{t_{i-1}t_{i+1}}{t_i^2}$ (for $i \ne 0, 6, 12$), and $a_i = \frac{t_{i-1}t_{i+1}s_i}{t_i^2}$, $b_i = \frac{t_i}{s_i^2}$ (for $i = 0, 6, 12$), so that $a_i^2 b_i = \frac{t_{i-1}^2 t_{i+1}^2}{t_i^3}$. Then these conditions are equivalent to asking for a_i, $(i \ne 0, 6, 12)$ and $a_i^2 b_i$ $(i = 0, 6, 12)$ to be regular functions of t or, equivalently, for a_i to be regular functions of t and $b_0 = b_6 = b_{12} = 1$.

Similarly, the entire 16-dimensional family can be defined parametrically as follows: Let A be the monomial subalgebra of $\mathbb{C}[t_i, t_i^{-1}]$ that is the normalization of the subalgebra generated by $a_i = \frac{t_{i-1}t_{i+1}}{t_i^2}$ $(i \ne 0, 6, 12)$, $a_i^2 b_i = \frac{t_{i-1}^2 t_{i+1}^2}{t_i^3}$ $(i = 0, 6, 12)$. Then the family is $\operatorname{Proj} R \to \operatorname{Spec} A$, where R is the subalgebra of the algebra $A[x,y,z]$, graded by $\deg x = \deg y = \deg z = 1$, generated by the monomials

$$u_c = x^2y^2z^2,\ u_0 = t_0x^6,\ u_1 = t_1x^5y,\ \dots,\ u_{17} = t_{17}x^5z.$$

More precisely, one may have to make a ramified finite base change $t_i = (t_i')^m$ for some divisible m to obtain a monomial ring $A_1 \supset A$, and then divide by $(\mu_m)^{16}$ to obtain a family of stable pairs over a stack $[\operatorname{Spec} A_1 : (\mu_m)^{16}]$. In our case, taking $m = 2$ suffices.

A similar construction holds for the 3-3-12 triangle shown in Figure 4. In this case R is the subring of the ring $\mathbb{C}[t][u,v,w]$, graded by $\deg u = \deg w = 1$, $\deg v = 4$, which is generated by the monomials

$$u_c = u^2v^2w^2,\ u_0 = t^{T_0}u^{12},\ u_1 = t^{T_1}u^8v,\ \dots,\ u_{17} = t^{T_{17}}u^{11}w.$$

The conditions for the heights to define a tiling $\cup Q_i$ that is a coarsening of the triangulation $\mathcal{T}_0$ are now

$$T_{i-1} + T_{i+1} - 2T_i \geq 0\ (i \neq 0,3,6),$$
$$T_{i-1} + T_{i+1} - \tfrac{3}{2}T_i \geq 0 \iff T_{i-1} + T_{i+1} + S_i - 2T_i \geq 0,\ S_i = \tfrac{1}{2}T_i\ (i = 0,6),$$
$$T_2 + T_4 \geq 0 \iff T_2 + T_4 + 2R_3 - 2T_3 \geq 0,\ R_3 = T_3.$$

Letting $t_i = t^{T_i}$, $s_i = t^{S_i}$ and $r_i = t^{R_i}$ as before, denote $a_i = \frac{t_{i-1}t_{i+1}}{t_i^2}$ (for $i \neq 0,3,6$), denote $a_i = \frac{t_{i-1}t_{i+1}s_i}{t_i^2}$, $b_i = \frac{t_i}{s_i^2}$ (for $i = 0,6$), and denote $a_3 = \frac{t_2t_4r_3^2}{t_3^2}$, $d_3 = \frac{t_3^2}{r_3^2}$. We obtain $a_i^2b_i = \frac{t_{i-1}^2t_{i+1}^2}{t_i^3}$ (for $i = 0,6$) and $a_3d_3 = t_2t_4$. So these conditions are equivalent to asking for a_i, $(i \neq 0,3,6)$, $a_i^2b_i$ $(i = 0,6)$ and a_3d_3 to be regular functions of t or, equivalently, for a_i to be regular functions of t and $b_0 = b_6 = d_3 = 1$.

Finally, we may define the entire 16-dimensional family parametrically as follows: Let A be the monomial subalgebra of $\mathbb{C}[t_i, t_i^{-1}]$ that is the normalization of the subalgebra generated by $a_i = \frac{t_{i-1}t_{i+1}}{t_i^2}$ $(i \neq 0,6)$, $a_i^2b_i = \frac{t_{i-1}^2t_{i+1}^2}{t_i^3}$ $(i = 0,3,6)$, and $a_3d_3 = t_2t_4$. Then the family is $\operatorname{Proj} R \to \operatorname{Spec} A$, where R is the subalgebra of the algebra $A[u,v,w]$, graded by $\deg u = \deg w = 1$, $\deg v = 4$, generated by the monomials

$$u_c = u^2v^2w^2,\ u_0 = t_0u^{12},\ u_1 = t_1u^8v,\ \dots,\ u_{17} = t_{17}u^{11}w.$$

4.4. **All surfaces claimed in Theorem 4.1 occur as stable limits.** The first step towards proving Theorem 4.1 is to show that all of the claimed surfaces actually occur as stable limits. We begin our proof by reducing the number of subdiagrams that we need to consider. Indeed, we have:

Lemma 4.14. *Let $\Gamma' \subset \Gamma$ be subdiagrams of Γ_{Vin}, with Γ maximal parabolic or elliptic, and Γ' elliptic. Suppose that the surface corresponding to Γ occurs as a stable limit. Then the surface corresponding to Γ' also occurs as a stable limit.*

Proof. Let $X = \cup X_s$ (resp. X') be a surface corresponding to the diagram Γ (resp. Γ') as in Theorem 4.1. According to the Rules 4.2, X is glued from toric surfaces X_s along toric boundaries, with a single exception as in 4.2(5). We first note that for each diagram Γ the surface X can be glued from X_s uniquely, so there is only one isomorphism class. (Of course, the pair (X,B) is not unique.)

We show that for each irreducible component X_s there exists a family $\pi_i\colon \mathcal{X}_s \to \mathbb{A}^1$ such that a fiber $\pi_t^{-1}(t)$ for $t \neq 0$ is isomorphic to X_s and $\pi_t^{-1}(0)$ is isomorphic to a union of toric surfaces $\cup X'_{ss'}$ which give a part of the surface X'. By gluing, this gives a family $\pi\colon \mathcal{X} \to \mathbb{A}^1$ such that for $t \neq 0$ the fiber $\pi^{-1}(t) \simeq X$ and $\pi^{-1}(0) \simeq X'$. A limit of smoothable surfaces is smoothable (by using a "diagonal" argument), so this proves the statement.

Once we know that X' is smoothable via some family $\pi'\colon \mathcal{X}' \to S$, $X' = \pi'^{-1}(0)$, where S is a smooth affine curve, it follows that the family of *pairs*

(X', B') is also smoothable. Indeed, on every fiber of π' the higher cohomologies of $\mathcal{O}(-2K_{X'})$ vanish. By the Theorem on Cohomology and Base Change, this implies that $\pi'_*\mathcal{O}(-2K_{\mathcal{X}'})$ is a locally free sheaf (of rank 28). Thus, the homomorphism $H^0(\mathcal{X}', \mathcal{O}(-2K_{\mathcal{X}'}) \to H^0(X', \mathcal{O}_{X'}(-2K_{X'}))$ is surjective, and any section $B' \in |-2K_{\mathcal{X}'}|$ can be lifted to a family of divisors $\mathcal{B}'$.

Thus, we are reduced to constructing a degeneration $X_s \rightsquigarrow \cup X'_{ss'}$ from an irreducible component X_s of X that corresponds to a connected elliptic or parabolic diagram Γ_s to a union of components $\cup X'_{ss'}$ corresponding to its subdiagram Γ'_s. We split the inclusion $\Gamma'_s \subset \Gamma_s$ into two steps:

(1) Removing some of the vertices b_i, d_i from Γ_s: this step smooths some corners and does not increase the number of irreducible components.
(2) Removing some of the a_i vertices: this step smooths some of the edges going from the center and increases the number of irreducible components, unless $\Gamma_s = \widetilde{A}_{17}$ and $\Gamma'_s = A_{17}$.

Step 1. First let Γ_s be parabolic. When $\Gamma_s = \widetilde{A}_1 \supset \Gamma'_s = A_1$, the degeneration takes $\mathbb{P}^2$ to itself and the double curve degenerates from a smooth conic to a pair of lines. The case $\widetilde{E}_7 \supset A_7$ also corresponds to a degeneration of $\mathbb{P}^2$, this time to a cone over a rational normal curve of degree four $\mathbb{P}^2 \rightsquigarrow \mathbb{F}_4^0$.

The case $\widetilde{D}_{10} \supset D_{10}$ is the degeneration $\mathbb{F}_1 \rightsquigarrow \mathbb{F}_3$ and $\widetilde{D}_{16} \supset D_{16}$ is $\mathbb{F}_4 \rightsquigarrow \mathbb{F}_6$. Both of these are special cases of a well-known degeneration $\mathbb{F}_n \rightsquigarrow \mathbb{F}_{n+2}$ for any n, see e.g. [MK71, p.26] or [BHPvdV04, Section VI.8]. Finally, the case $\widetilde{E}_8 \supset A_8$ is the degeneration $\mathbb{F}_2^0 = \mathbb{P}(1,1,2) \rightsquigarrow \mathbb{P}(1,2,9)$ constructed as a degeneration of a general cubic hypersurface in $\mathbb{P}(1,1,2,3)$ to a cubic not involving the degree 3 variable. The embedding $\mathbb{P}(1,2,9) \hookrightarrow \mathbb{P}(1,1,2,3)$ is the Veronese-type embedding $(x,y,z) \mapsto (x^3, xy, y^3, z)$.

Let now Γ_s be elliptic. We can assume that Γ_s is not a single vertex d_i because we ignore these by Rule 4.2(6). Then X is a surface with $\rho = 2$ or 3, cf. Table 3. If $\rho = 2$ then Γ_s is D_n, E_n $(n = 6,7,8)$, or A'_n, and $\Gamma'_s = A_{n-1}$. The surface X_s is a toric surface corresponding to a 4-gon, and X'_s corresponds to a triangle obtained from it by "smoothing a corner". We think of the diagram Γ_s as a fork centered at the vertex a_0 with three legs of lengths p, q, r, so that $q = 2$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. One has $p = 1$ for A'_n, $p = 2$ for D_n, and $p = 3$ for E_n. Γ'_s is obtained from Γ_s by removing the vertex b_0 from the q-leg.

The case $D_n \rightsquigarrow A_{n-1}$ with even n is again an application of the standard degeneration $\mathbb{F}_{n/2-2} \rightsquigarrow \mathbb{F}_{n/2}$. In all cases the following degeneration works. Consider the generators u_i, u_c in the graded ring $k[x^{\pm1}, y^{\pm1}, z^{\pm1}][t^{\pm1}]$ defined by:

$$u_0 = x^3, \quad u_c = xyz - \frac{1}{t}x^3,$$
$$u_1 = x^5y,\ u_2 = x^2y,\ , u_3 = x^3y^3, \dots \quad u_{17} = tx^5z,\ u_{16} = tx^2z,\ u_{15} = t^3x^3y^3.$$

The grading is given by $\deg x = \deg y = \deg z = 1$, $\deg t = 0$, and $\deg u_c = 3$ (resp. 6) for i even (resp. odd). Let R be a $k[t]$ subalgebra generated by the u_i's for which the vertices a_i are present in Γ_s. Let $\mathcal{X}_s = \operatorname{Proj} R$ and let $\pi_s \colon \mathcal{X}_s \to \mathbb{A}^1 = \operatorname{Spec} k[t]$ be the induced map. We have the following relations. For $i \in \mathbb{Z}_{18}$ even, $i \neq 0$: $u_i^4 = u_{i-1}u_{i+1}$; for i odd: $u_i = u_{i-1}u_{i+1}$; plus the relation

$$(tu_c + u_0)u_0^3 = u_1 u_{17}.$$

For $t \neq 0$ they define a toric surface X for the 4-gon determined by Γ_s. For $t = 0$, the last relation becomes $u_0^4 = u_1 u_{17}$ and the a_0-corner is "smoothed out". The central fiber is X' for the diagram Γ'_s.

Finally, let X be a surface with $\rho = 3$, corresponding to a 5-gon. Then Γ is one of D'_{10}, D'_{16}, A''_9, A''_{15}. We use a similar parametric degeneration to the above, with the following changes which we illustrate in the case of D'_{10}:

$$u_c = xyz - \frac{1}{t}x^3 - \frac{1}{t}y^3,\ u_0 = x^3,\ u_1 = x^5 y,\ u_2 = x^2 y, \dots,\ u_6 = y^3,$$
$$u_{17} = tx^4(xz - \frac{1}{t}y^2)^2,\ u_{16} = tx(xz - \frac{1}{t}y^2),\ u_7 = ty^4(yz - \frac{1}{t}x^2)^2.$$

A computation is similar to the one above.

Step 2. This part is easy: this is simply a toric degeneration of a toric surface X_s into a union of toric surfaces $\cup X'_{ss'}$ corresponding to a subdivision of the polytope Q_i of X_s into subpolytopes $\cup Q'_{ss'}$ by edges from the center of the big triangle. □

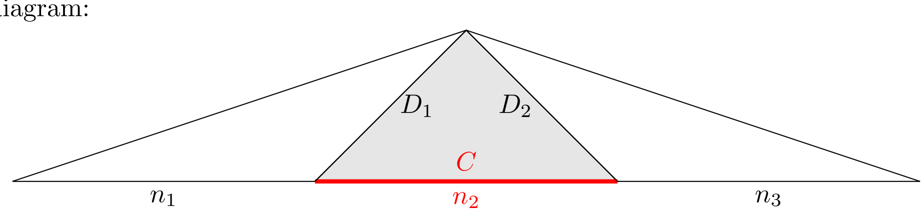
Using this, we may reduce our problem to that of proving that the surfaces corresponding to the *maximal subdiagrams* of Γ_{Vin} occur as stable limits. A maximal subdiagram is an elliptic or parabolic subdiagram of Γ_{Vin} that has rank 18. As every subdiagram $\Gamma \subset \Gamma_{\text{Vin}}$ is contained in a maximal one, by Lemma 4.14, this is enough to prove that the stable limits corresponding to all subdiagrams exist.

It follows from Lemma 2.4 that, up to equivalence 4.6, there are 103 maximal subdiagrams of Γ_{Vin}, 99 of which are elliptic and 4 of which are parabolic. They are listed in the table in the appendix to this paper. As may be seen from that appendix, most of the stable limits corresponding to these maximal subdiagrams may be constructed by the toric method of Section 4.2; indeed, any case labeled "Toric" may be constructed from a subdivision of the 6-6-6 triangle, and any case labelled "Unigonal" may be constructed from a subdivision of the 3-3-12 triangle.

Most of the remaining cases are constructed by the following method. We begin by constructing a carefully chosen (usually non-maximal) 1-parameter degeneration $\mathcal{X} \to \mathbb{A}^1$, using the toric method of Section 4.2. Then we perform a number of birational *elementary modifications*, as described below, to obtain a new family $\mathcal{X}' \to \mathbb{A}^1$ with a new central fiber of the required type.

The singularities of the pair $\big(\mathcal{X}', (\frac{1}{2} + \epsilon)B\big)$ are made worse by this operation. But we may pick a new divisor B' by taking a generic section of $|-2K_{\mathcal{X}'}|$, as in the proof of 4.14, so we only need pay attention to the structure of the central fiber $\mathcal{X}'_0$.

4.4.1. *Elementary modifications of type 1.* Suppose there is a toric 1-parameter degeneration $(\mathcal{X}, \mathcal{L}) \to S$ in which part of the central fiber appears as in the following diagram:



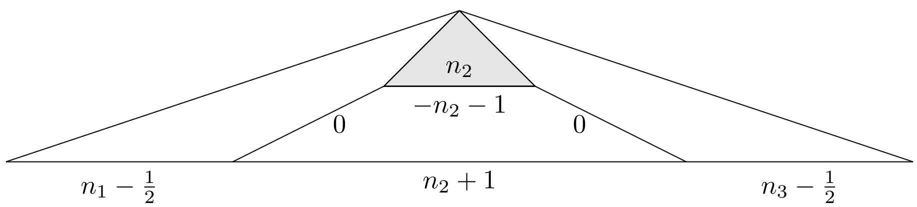
Label the three surfaces in this diagram Y_1, Y_2, Y_3 from left to right; they can be any of the surfaces Y allowed by Remark 4.8. The red marked curve is a generic smooth rational curve linearly equivalent to $-K_{Y_2} - D$ (where $D = D_1 + D_2$ denotes the double curve on Y_2), we denote it by C. The n_i are half-integers indicating self-intersections of curves.

Assume that the central fiber $X = \mathcal{X}_0$ is reduced and that the threefold total space $\mathcal{X}$ satisfies the following assumption on singularities:

Assumption 4.15. The only singularities in a neighborhood of C in $\mathcal{X}$ lie on the double curves D_i that meet C. They have the following form:

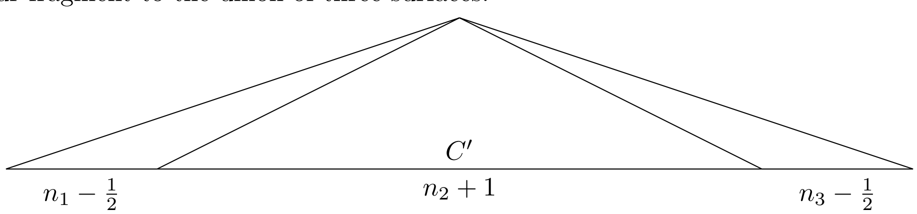
- If D_i is a "short" side (i.e. if Y_2 contains a $\frac{1}{2}(1,1)$ singularity at the intersection $C \cap D_i$), then $\mathcal{X}$ has an isolated $\frac{1}{2}(1,1,1)$ singularity at $C \cap D_i$ and is smooth at the generic point of D_i.
- If D_i is a "long" side (i.e. if Y_2 is smooth at the intersection $C \cap D_i$), then $\mathcal{X}$ has an A_1 singularity at the generic point of D_i.

Now perform a blow-up of $\mathcal{X}$ along C. We have the following diagram:



Indeed, this is an easy toric computation: the blow-up is simply obtained by cutting an edge.

Let E be the exceptional divisor and let $\mathcal{L}' = \mathcal{L} - 2E$. Then $\mathcal{L}'$ is nef and it restricts to the strict transform of Y_2 as zero. The line bundle $\mathcal{L}'$ defines a contraction to the surface which equals X outside of our fragment, and contracts our fragment to the union of three surfaces.



Indeed, we claim that any relatively nef line bundle on $\mathcal{X}$ is semiample. Since the central fiber $\mathcal{X}_0$ has slc singularities, by Inversion of Adjunction [Kaw07] the pair $(\mathcal{X}, \mathcal{X}_0)$ is log canonical in a neighborhood of $\mathcal{X}_0$, thus $\mathcal{X}$ is canonical. One easily sees that $-K_{\mathcal{X}}$ is relatively ample by restricting to the irreducible components of $\mathcal{X}_0$. Now the Base Point Free Theorem (see e.g. [KM98, Thm.3.3]) applies.

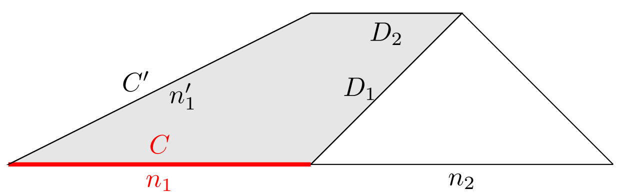
If $n_i > \frac{1}{2}$ for both $i \in \{1,3\}$, then the curve C' in the new threefold $\mathcal{X}'$ also satisfies Assumption 4.15 and we may repeat the process if necessary. If $n_i = \frac{1}{2}$, for $i = 1,3$, the corresponding surface Y_i also gets contracted; after this contraction has been performed Assumption 4.15 is no longer satisfied, so we cannot modify this component further.

The effect of this elementary modification upon the subdiagram corresponding to the component Y_2 is given in Table 4; it corresponds to the subdiagram obtained by deleting any internal (b_i, d_i) vertices and extending the remaining diagram by one vertex in each direction. The subdiagrams corresponding to the two neighboring components lose one outer (a_i) vertex each.

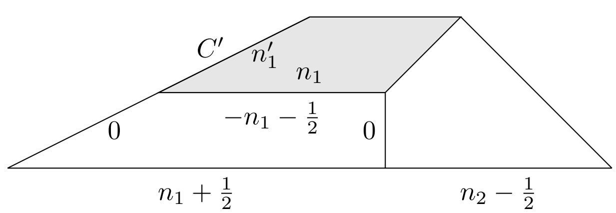
Case	Subdiag.	Elementary modification	
		Type 1	Type 2
1	A^e_{2n-1}	A^c_{2n+1}	n/a
2	A_{2n}	A_{2n+2}	n/a
3	A^c_{2n-1}	A^e_{2n+1}	n/a
4	D_{2n}	A^c_{2n+1}	D_{2n+1}
5	D_{2n-1}	A_{2n}	D_{2n}
6	A'_{2n-1}	A_{2n}	A'_{2n}
7	A'_{2n}	A^e_{2n+1}	A'_{2n+1}
8	E_7	A_8	n/a
9	E_8	A^c_9	n/a
10	E_6	A^c_7	n/a
13	D'_{2n}	A_{2n}	D_{2n}, A'_{2n}
14	A''_{2n-1}	A^c_{2n-1}	A'_{2n-1}
	$\tilde{A}_1$	A^c_3	n/a
	$\tilde{E}_7$	A^c_9	n/a
	$\tilde{D}_{10}$	A^c_{11}	D_{11}
	$\tilde{D}_{16}$	A^c_{17}	D_{17}

TABLE 4. Effect of elementary modifications on subdiagrams.

4.4.2. *Elementary modifications of type 2.* Now, in a completely similar way, consider a toric 1-parameter degeneration $(\mathcal{X}, \mathcal{L}) \to S$ in which part of the central fiber appears as follows:

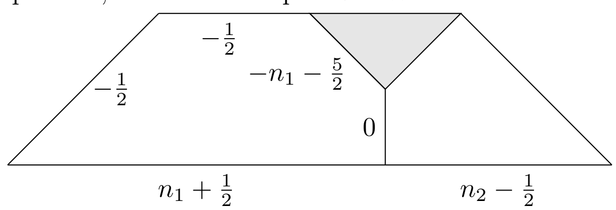


The shaded surface on the left is a toric surface Y with Picard rank ≥ 2. The red curve C is an irreducible rational curve, so that $C + C'$ is linearly equivalent to $-K_Y - D$ (where $D = D_1 + D_2$ denotes the double curve on Y_2), for some effective divisor C'. As before, the n_i, n'_i are half-integers giving intersection numbers of curves; here we assume that $n'_1 \in \{0, -\frac{1}{2}\}$ (which implies that Y does not correspond to a subdiagram of type E_n). We further assume that the central fiber $X = \mathcal{X}_0$ is reduced and that the singularities of the threefold total space $\mathcal{X}$ in a neighborhood of C satisfy Assumption 4.15. As before, we perform a blow-up along the curve C to obtain a new family $\mathcal{X}' \to S$ as follows.

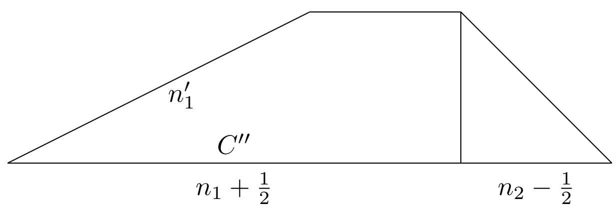


Let E be the exceptional divisor and let $\mathcal{L}' = \mathcal{L} - 2E$. If $n_1' = 0$, then $\mathcal{L}'$ is nef and restricts to the strict transform of Y as a ruling. If $n_1 = -\frac{1}{2}$, then we need to perform a flip along the curve C'.

The existence of this operation can be proved as follows. Consider the double cover $\mathcal{Y} \to \mathcal{X}$ ramified along a generic divisor in $|-2K_{\mathcal{X}}|$. The curve C' is part of the fixed locus of the linear system $|-2K_{\mathcal{X}}|$, where it appears with multiplicity 2. Thus, the central fiber of $\mathcal{Y}$ has a double curve with self-intersection $(-1,-1)$ over C', and $K_{\mathcal{Y}}$ is numerically trivial along this double curve by adjunction. Therefore, since flops exist for threefolds [KM98, Section 6], we may flop this double curve. Taking the $\mathbb{Z}_2$ quotient, we obtain a flip of C'. We have the following picture.



After performing this flip, the line bundle $\mathcal{L}'$ becomes nef and trivial on the strict transform of Y. Thus, for both $n_i' \in \{0, -\frac{1}{2}\}$ the line bundle $\mathcal{L}'$ defines a contraction to the surface which equals X outside of our fragment, and contracts our fragment to the union of two surfaces:



As before, if $n_2 > \frac{1}{2}$, the curve C'' in the new threefold $\mathcal{X}'$ also satisfies Assumption 4.15, so we may repeat the process if necessary. If $n_2 = \frac{1}{2}$, then the right-hand surface gets contracted; after this contraction has been performed Assumption 4.15 is no longer satisfied, so we cannot modify this component further.

The effect of this elementary modification upon the subdiagram corresponding to the component Y is given in Table 4; note that in case 13 there are two possibilities, corresponding to different choices of curve C. The subdiagram corresponding to the neighboring component on the right loses one outer (a_i) vertex.

Remark 4.16. We note that the assumptions required to perform these elementary modifications are also satisfied by the Type II degenerations corresponding to the parabolic subdiagrams $\tilde{D}_{10}\tilde{E}_7$ and $\tilde{D}_{16}\tilde{A}_1$ (in which case D_1 and D_2 represent the two "halves" of the unique double curve D). In particular, if we take a smooth degeneration to one of these surfaces, after performing a base change we can arrange for the threefold $\mathcal{X}$ to have a curve of A_1 singularities along the double curve D, so Assumption 4.15 is satisfied. The result of performing elementary modifications upon the components in these Type II degenerations is given by the last four lines in Table 4.

4.4.3. *Constructing maximal subdiagrams.* We now use these operations to construct stable limit surfaces corresponding to maximal subdiagrams. In many cases, such stable limits may be constructed by starting from a Type II degeneration

$\tilde{D}_{10}\tilde{E}_7$ or $\tilde{D}_{16}\tilde{A}_1$ and performing a series of elementary modifications; these cases are detailed in the table in the Appendix.

This leaves 13 cases. Most can be constructed by starting with a carefully chosen 1-parameter degeneration $\mathcal{X} \to \mathbb{A}^1$, built using the toric method of Section 4.2, then performing elementary modifications to get the required degeneration.

All of the toric degenerations that we use are constructed from the 6-6-6 triangle. As in Section 4.3, to define such a toric degeneration, we need to specify a set of heights $\{h_0, \dots, h_{17}\}$ associated to the points $\{a_0, \dots, a_{17}\}$ from Figure 3b (the central point is always taken to have height 0). These heights must be chosen so that Assumption 4.15 is satisfied in the neighbourhood of any curve C that will be used in an elementary modification. This assumption may be checked using the following lemmas.

For concreteness, choose a system of coordinates for $\mathbb{Z}^2$ in which the center has coordinates $(0, 2)$ and the edge C_{01} to be modified along has end points $(m_0, 0)$ and $(m_1, 0)$.

Lemma 4.17. *The curve C_{01} corresponding to this edge is smooth (resp. has an A_1 singularity along it) iff the distance d_{01} from the segment $(m_0, h_0) - (m_1, h_1)$ to the origin (or to any point $(2a, 2b)$) is even (resp. odd).*

Proof. The points $(m_i, 0, h_i)$ are integral vectors in the lattice $M \simeq \mathbb{Z}^3$. We compute a part of the normal fan in the dual lattice $N \simeq \mathbb{Z}^3$. We compute a vector perpendicular to the facet through the points $(0, 2, 0)$, $(m_0, 0, h_0)$, $(m_1, 0, h_1)$ to be

$$v_{01} = \big(2(h_0 - h_1),\ m_0h_1 - m_1h_0,\ 2(m_1 - m_0)\big)$$

A vector perpendicular to the vertical facet, corresponding to the second toric divisor containing the curve C_{01} is $(0, 1, 0)$. Let u_{01} be a primitive vector proportional to v_{01}. Then the condition for $\mathcal{X}$ to be smooth along C_{01} is that the vectors u_{01} and $(0, 1, 0)$ generate a cotorsion-free sublattice of N. This is seen to be equivalent to the following condition:

$$\mathrm{GCD}(2(h_1 - h_0), 2(m_1 - m_0)) = \mathrm{GCD}(2(h_1 - h_0), 2(m_1 - m_0), m_0h_1 - m_1h_0)$$

Let $G = \mathrm{GCD}(h_1 - h_0, m_1 - m_0)$. Then G is the lattice length of the interval $(m_0, h_0) - (m_1, h_1)$, and $m_0h_1 - m_1h_0$ is the lattice area of the triangle with base on the above interval and height d_{01}; it equals Gd_{01}. Thus the above condition holds iff d_{01} is even.

If d_{01} is odd then the two vectors generate a sublattice with cotorsion group $\mathbb{Z}_2$, and $\mathcal{X}$ has generically an A_1 singularity along C. □

Next, we consider two edges $(m_0, h_0) - (m_1, h_1)$ and $(m_1, h_1) - (m_2, h_2)$ and the singularity along the curve D which is the intersection of the corresponding irreducible components in the central fiber $\mathcal{X}_0$.

Notation 4.18. Let $\bar{v}_{ij} = \big(2(h_i - h_j), 2(m_j - m_i)\big) \in \mathbb{Z}^2$ and let $\bar{u}_{ij}$ be a primitive vector in $\mathbb{Z}^2$ proportional to it. Let d_{ij} denote the distance from the segment $(m_i, h_i) - (m_j, h_j)$ to the center. Finally, let D_i denote the edge from the center to $(m_i, 0, h_i)$ and let P_i denote the point corresponding to the end point of this edge.

Lemma 4.19. *If the vectors $\bar{u}_{01}, \bar{u}_{12}$ span $\mathbb{Z}^2$ then the 3-fold $\mathcal{X}$ is smooth along D_1 and at the point P_1.*

If the vectors $\bar{u}_{01}, \bar{u}_{12}$ span a sublattice of index 2 in $\mathbb{Z}^2$ then:

- *If $\frac{1}{2}d_{01} + \frac{1}{2}d_{12}$ is odd, then $\mathcal{X}$ is smooth along D_1 and has a $\frac{1}{2}(1,1,1)$ singularity at the point P_1.*
- *If $\frac{1}{2}d_{01} + \frac{1}{2}d_{12}$ is even, then $\mathcal{X}$ has an A_1 singularity along D_1.*

Proof. One has

$$\mathbb{Z}^3/\langle u_{01}, u_{12}, (0,1,0)\rangle = \mathbb{Z}^2/\langle \bar{u}_{01}, \bar{u}_{12}\rangle$$

and the first half follows. For the second half: two primitive vectors generate a sublattice of index 2 in $\mathbb{Z}^2$ iff $\bar{u}_{01} + \bar{u}_{12} = 2\bar{w}$ for some $\bar{w} \in \mathbb{Z}^2$. The condition we are after is that $u_{01} + u_{10} \neq 2w$ for some $w \in \mathbb{Z}^3$. This means that the second coordinate $\frac{1}{2}d_{01} + \frac{1}{2}d_{12}$ should be odd. □

Corollary 4.20. *Suppose that $m_1 - m_0 = m_2 - m_1 = 1$. Then:*

(1) *The 3-fold X is smooth along the outside edges C_{01}, C_{12} iff the heights h_i are even for the even numbers m_i among m_0, m_1, m_2.*
(2) *If m_1 is even then X has an A_1 singularity along $D_1 \iff h_2 - 2h_1 + h_0 = 2$.*
(3) *If m_1 is odd then X is smooth along $D_1 \iff X$ has a $\frac{1}{2}(1,1,1)$ singularity at $P \iff h_2 - 2h_1 + h_0 = 2$.*

At the edges corresponding to the corners the computation is similar. We give some partial results:

Lemma 4.21. *In the 6-6-6 triangle, suppose that there is an edge at the corner a_0 and the lengths of the adjacent edges are both 1. Then the 3-fold has an A_1 singularity along the edge $D_0 \iff h_0$ is even and $h_{17} - \frac{3}{2}h_0 + h_1 = 2$.*

Using this, we give constructions of for all but one of the missing stable limits corresponding to maximal subdiagrams in Table 5. In this table, "#" gives the number of the maximal subdiagram, using the enumeration from the table in the Appendix, "Subdg." gives the corresponding maximal subdiagram, "Starting Degen." gives the subdiagram corresponding to the toric degeneration we begin with, which is defined by assigning "Heights" to the points $\{a_0, \dots, a_{17}\}$ (unlisted heights may be obtained as appropriate linear combinations of listed ones), and "Construction" gives the sequence of elementary modifications that must be performed on the starting degeneration to obtain the required one.

4.4.4. *Case 6.* There is one stable limit corresponding to a maximal subdiagram that we have yet to construct: case 6 from the table in the Appendix, corresponding to the subdiagram $D'_{16}A_2$. It can be shown that this stable limit cannot be obtained by elementary modifications from any toric degeneration constructed using the 6-6-6 triangle. Moreover, if one attempts to construct this degeneration using the 3-3-12 triangle, Assumption 4.15 is not satisfied, so the elementary modification process is not well-defined. We therefore have to construct this degeneration by a different method.

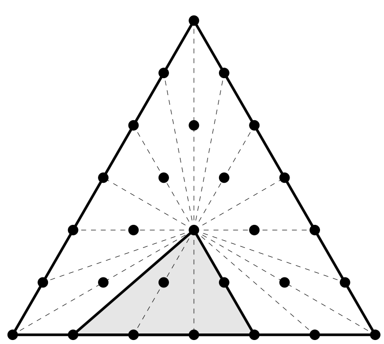
We consider each component of this subdiagram in turn. Start with the A_2. This corresponds to an irreducible component of a stable pair that may be constructed in the 6-6-6 triangle, as shown in Figure 6a. As described in Section 4.3, we may write a system of parametric equations for such a component:

$$u_c = x^2y^2z^2,\ u_1 = t^{T_1}x^5y,\ u_2 = t^{T_2}x^4y^2,\ u_3 = t^{T_3}x^3y^3,\ u_4 = t^{T_4}x^2y^4,$$

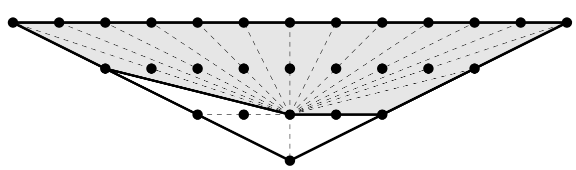
where (x, y, z) are variables of degree 1 and the T_i satisfy the equations $T_{i+1} - 2T_i + T_{i-1} = 0$ induced from their relative positions in the polytope from Figure 6a.

#	Subdg.	Starting Degen.	Heights	Construction
11	$A'_{13}A_4$	$A''_9A_4A_4$	$(h_0,h_6,h_7,h_{12},h_{17})$ $=(0,0,0,10,0)$	5 type 2 mod. on A''_9
13	$A_{14}A_2$	$D_5A'_5A_4A_2$	$(h_0,h_4,h_6,h_9,h_{12},h_{17})$ $=(0,0,4,4,10,0)$	5 type 1 mod. on A'_5
16	$A'_{12}A'_6$	$A''_9A^c_3A^c_3A'_2$	$(h_0,h_6,h_7,h_{11},h_{12},h_{13},h_{17})$ $=(0,0,0,8,12,8,0)$	4 type 2 mod. on A''_9 4 type 2 mod. on A'_2
18	$A^c_{13}A'_4$	$A'_6A'_6A^c_3A'_2$	$(h_0,h_5,h_6,h_{11},h_{12},h_{13},h_{17})$ $=(0,0,2,7,10,8,0)$	2 type 2 mod. on A'_2 4 type 1 mod. on A'_6
21	$A'_{11}D_7$	$A''_9D_7A_2$	$(h_0,h_6,h_7,h_{10},h_{12},h_{17})$ $=(0,0,0,6,15,0)$	3 type 2 mod. on A''_9
23	$A_{12}D_5$	$A'_7D_5A_2A_2$	$(h_0,h_6,h_9,h_{12},h_{14},h_{17})$ $=(0,0,6,15,6,0)$	3 type 1 mod. on A'_7
26	$A'_{10}E_8$	$A''_9E_8A^c_1$	$(h_0,h_6,h_7,h_9,h_{12},h_{17})$ $=(0,0,0,4,40,0)$	2 type 2 mod. on A''_9
27	$A^c_9E_8$	$E_8A'_6A^c_1A^c_1$	$(h_0,h_5,h_6,h_7,h_{12},h_{15},h_{17})$ $=(0,0,2,3,58,4,0)$	2 type 1 mod. on A'_6
29	$A_{10}E_7$	$E_7A'_7A^e_1A^c_1$	$(h_0,h_6,h_8,h_{12},h_{15},h_{17})$ $=(0,0,4,28,4,0)$	2 type 1 mod. on A'_7
33	$D'_{10}A_8$	$D'_{10}A'_3A_2A_2$	$(h_0,h_1,h_4,h_6,h_{12},h_{13},h_{16})$ $=(0,0,6,36,18,6,0)$	3 type 1 mod. on A'_3
50	$A_{10}A'_7$	$A'_7A^e_3A^c_3A'_3$	$(h_0,h_2,h_6,h_{12},h_{13},h_{17})$ $=(0,0,8,14,8,0)$	4 type 1 mod. on A'_3
51	$A^e_9D_8$	$D_8A'_4A_2A_2$	$(h_0,h_3,h_6,h_{12},h_{14},h_{17})$ $=(0,0,6,24,6,0)$	3 type 1 mod. on A'_4

TABLE 5. Construction of stable limits corresponding to maximal subdiagrams



(A) A_2 component



(B) D'_{16} component

FIGURE 6. Toric components of $D'_{16}A_2$

Next, consider the D'_{16}. This corresponds to an irreducible component of a stable pair that may be constructed in the 3-3-12 triangle, as shown in Figure 6b. As above, we may write a system of parametric equations for such a component:

$$u_c = u^2v^2w^2, \quad u_0 = t^{T_0}u^{12}, \quad u_1 = t^{T_1}u^8v, \quad u_4 = t^{T_4}v^2w^4, \quad u_5 = t^{T_5}vw^8,$$
$$u_6 = t^{T_6}w^{12}, \quad u_7 = t^{T_7}uw^{11}, \quad \ldots \quad u_{17} = t^{T_{17}}u^{11}w,$$

where (u, v, w) are variables of degree $(1, 4, 1)$ respectively and the T_i satisfy the equations

$$T_{i+1} - 2T_i + T_{i-1} = 0 \quad \text{for } i \notin \{0, 6\}$$
$$T_{i+1} - \frac{3}{2}T_i + T_{i-1} = 0 \quad \text{for } i \in \{0, 6\}$$

induced from their relative positions in the polytope from Figure 6b.

We wish to realize these two components as part of the same degeneration. To do this, we first need to embed the D'_{16} into a degeneration of $\mathbb{P}^2$'s, instead of $\mathbb{F}_4^0 = \mathbb{P}(1, 1, 4)$'s. This is achieved by performing a Veronese embedding of the central fiber

$$\mathbb{P}(1, 1, 4) \hookrightarrow \mathbb{P}(1, 1, 1, 2)$$
$$(u, w, v) \mapsto (x, y, z, s) = (u^2, w^2, uw, v)$$

and imposing the additional relation $t^n s = xy - z^2$. Putting this together, we obtain a system of parametric equations

$$u_c = z^2(xy - z^2)^2, \quad u_0 = t^{T_0}x^6,$$
$$u_1 = t^{T_1}x^4(xy - z^2), \quad u_4 = t^{T_4}y^2(xy - z^2)^2, \quad u_5 = t^{T_5}y^4(xy - z^2),$$
$$u_6 = t^{T_6}y^6, \quad \ldots \quad u_{12} = t^{T_{12}}z^6, \quad \ldots \quad u_{17} = t^{T_{17}}x^5z.$$

Note here that the T_i in this equation are not necessarily the same as the T_i we had before; they may have been modified by adding/subtracting multiples of n. However, after performing a linear rescaling we may assume that they satisfy the same set of relations, so we abuse notation and use the same symbols to denote them.

Now we need to glue this component together with the A_2 component we had before. To do this, we have to make the parametric equations for u_c, u_1, u_4 agree. This is simply achieved by making the substitution $xy \mapsto (xy - z^2)$ in the parametric equations for A_2.

Thus, we have a system of parametric equations:

$$u_c = z^2(xy - z^2)^2,$$
$$u_0 = t^{T_0}x^6, \quad u_1 = t^{T_1}x^4(xy - z^2), \quad u_2 = t^{T_2}x^2(xy - z^2)^2,$$
$$u_3 = t^{T_3}(xy - z^2)^3, \quad u_4 = t^{T_4}y^2(xy - z^2)^2, \quad u_5 = t^{T_5}y^4(xy - z^2),$$
$$u_6 = t^{T_6}y^6, \quad u_7 = t^{T_7}y^5z, \quad \ldots$$
$$u_{12} = t^{T_{12}}z^6, \quad \ldots \quad u_{17} = t^{T_{17}}x^5z.$$

Here the T_i satisfy the relations as given above. It just remains to find a system of heights T_i satisfying these relations. One may see that

$$(T_0, \ldots, T_{12}, T_{13}, T_{14}, T_{15}, T_{16}, T_{17}) = (18, \ldots, 18, 9, 0, 3, 6, 9)$$

suffice.

Of course, this is not a proof. However, one may check the parametric equations given above using a computer algebra package (we used *Sage*) and see that they do indeed define a degeneration of $\mathbb{P}^2$'s to a stable limit of type $D'_{16}A_2$, as required. In fact, this is a very special case of a more general construction that will be discussed further in Section 6.

This completes the proof of the following theorem.

Theorem 4.22. *All of the surfaces claimed in Theorem 4.1 are smoothable to* $\mathbb{P}^2$*.*

4.5. **Only surfaces claimed in Theorem 4.1 occur as stable limits: preliminary reductions.** Let $(\mathcal{X}^*, \mathcal{B}^*) = (X_t = \mathbb{P}^2, (\frac{1}{2}+\epsilon)B_t) \to S \setminus 0$ be a 1-parameter degeneration of plane sextics. We would like to find its stable limit. First, we are going to make a number of simplifying reductions.

Lemma 4.23. *We can assume that the limit sextic* B_0 *in* $\mathbb{P}^2$ *is GIT-semistable and even more, that it lies is a closed* PGL_3*-orbit.*

Proof. Indeed, we get a map $\phi\colon S \to \overline{M}^{\mathrm{GIT}}$, since the latter variety is proper. Then $\phi(0)$ corresponds to a unique GIT-semistable B_0 whose orbit is closed. The map ϕ can be lifted, perhaps after a finite base change, to a map from S to the open set of GIT-semistable sextics which gives an isomorphic family over $S \setminus 0$. This new family is obtained from the old one by an element of $PGL_3\big(k((t))\big)$. □

At this point recall the classification of GIT-semistable sextics, with closed orbits:

(a) Stable.
(b) A triple conic.
(c) Properly semistable and not of type (b).

We consider each of these cases in turn. In case (a) the pair $(X_0, \frac{1}{2}B_0)$ is lc and, according to [Sha80, Thms 2.3, 2,4], the sextics for which the pair $(X_0, (\frac{1}{2}+\epsilon)B_0)$ is not lc are all of Type II and are as follows:

(a1) Double conic C_1 + conic C_2, such that $|C_1 \cap C_2| = 4$.
(a2) Double cubic.

It is easy to compute the stable limits in these cases: simply blow-up the double loci until $(X_0, (\frac{1}{2}+\epsilon)B_0)$ becomes lc, then contract any unstable components. The result is a type II degeneration corresponding to the subdiagram $\tilde{D}_{16}\tilde{A}_1$ in case (a1) and $\tilde{A}_{17}$ in case (a2).

In case (b), Shah [Sha80, Thms 4.3, 6.1] show that, after a possible further base change, we may assume that the limit is a curve of degree 12 in $\mathbb{P}(1,1,4) \cong \mathbb{F}_4^0$ that is either:

(b1) Stable, or
(b2) Properly semistable.

Moreover, in case (b1) the pair $(X_0, \frac{1}{2}B_0)$ is lc and, according to [Sha80, Thm 4.3], the only case where the pair $(X_0, (\frac{1}{2}+\epsilon)B_0)$ is not lc is that of a double conic Q_1 and single quartic Q_2, such that $|Q_1 \cap Q_2| = 4$. As before, this gives rise to a type II degeneration corresponding to the subdiagram $\tilde{D}_{16}\tilde{A}_1$.

Thus, we may assume that the limit (X_0, B_0) is of type (b2) or (c). In both cases, the pair $(X_0, \frac{1}{2}B_0)$ is lc. By the Hilbert-Mumford criterion for semistability, this means that for some choice of coordinates x, y, z on $\mathbb{P}^2$ (resp. $\mathbb{P}(1,1,4)$) the equation f of B_0 contains the monomial $x^2y^2z^2$ and all monomials lie on one side of a certain line passing through $(2,2,2)$.

Lemma 4.24. *Using these coordinates* x, y, z*, one can construct a toric degeneration* $(\mathcal{X}, \mathcal{B}) \to S$ *with the following properties:*

(1) *X_0 is a union of toric varieties Y_i corresponding to cutting the 6-6-6 triangle of sextic monomials in $\mathbb{P}^2$ (resp. the 3-3-12 triangle of degree 12 monomials in $\mathbb{P}(1,1,4)$) into several polytopes $\cup P_i$ from the center $(2,2,2)$ out to the border. Thus, every polytope P_i is 2-dimensional, contains $(2,2,2)$ on its border, and all the other vertices lie on the boundary of the big triangle.*
(2) *The divisor B_0 on X_0 is Cartier with an equation F, and the restriction F_i of this equation to Y_i has a Newton polytope (the convex hull of nonzero monomials) Q_i such that: Q_i is 2-dimensional, and Q_i has the same corner as P_i at the vertex $(2,2,2)$.*

Note that f is one of the equations F_i if the Newton polytope of f is two-dimensional.

Proof. Suppose first that the limit is of type (c). To prove the statement in this case, we look at the leading monomials in the equation $f_6 \in k[x,y,z][[t]]$ of the family of sextics, i.e. those monomials $c_{lmn}x^l y^m z^n$ that appear in the lower convex hull of the points

$$(l,m,n,\mathrm{val}(c_{lmn})) \in \mathbb{Z}^4.$$

This gives a polytopal decomposition of the 6-6-6 triangle which contains the polytopes Q_i near the point $(2,2,2)$. Then we extend Q_i to the boundary to obtain the polytopes P_i (note that the vertices on the boundary are automatically integral), and take the corresponding toric degeneration of $\mathbb{P}^2$.

Next suppose that the limit (X_0, B_0) is of type (b2). In this case the proof of [Sha80, Thm. 6.3] shows that (possibly after a further base change) we may embed our family $\mathcal{X}$ into $\mathbb{P}^5 \times S$ as a family of quadrics (on the generic fiber this is just a degree 2 Veronese embedding), and that the family of sextics $\mathcal{B}$ can be lifted to a family of cubic hypersurfaces $\mathcal{C}$ in $\mathbb{P}^5$. The central fiber X_0 in this family is a copy of $\mathbb{F}_4^0 \cong \mathbb{P}(1,1,4)$, embedded by a degree 4 Veronese embedding.

Now consider the family $X_0 \times S \subset \mathbb{P}^5 \times S$. The restriction of $\mathcal{C}$ to a generic fiber in this family is a divisor of degree 12 in $\mathbb{P}(1,1,4) \cong \mathbb{F}_4^0$, so we obtain a family of degenerating curves of degree 12 in $\mathbb{P}(1,1,4)$ whose limit is (X_0, B_0). Using this new family, the argument proceeds exactly as in the type (c) case above. □

Note that the monomials appearing in the equations F_i can be any of the 28 monomials $x^l y^m z^n$.

After these steps, we are reduced to classifying the pairs $(Y, \Delta + (\frac{1}{2} + \epsilon)B)$, where Y are toric varieties corresponding to the polytopes P as above, the Newton polytope of B is Q, and Δ is the boundary divisor corresponding to the sides of P through the vertex $(2,2,2)$. In type II, Δ is irreducible, and in type III, $\Delta = \Delta_1 + \Delta_2$ has two components. The polytope P corresponds to a polarized toric variety (Y, L), and $\mathcal{O}_Y(B) \simeq L$.

In type II, there are 3 possible P's in the 6-6-6 triangle, corresponding to the parabolic subdiagrams $\tilde{E}_7$, $\tilde{E}_8$, and $\tilde{D}_{10}$, and 3 possible P's in the 3-3-12 triangle, corresponding to the parabolic subdiagrams $\tilde{A}_1$, $\tilde{E}_8$, and $\tilde{D}_{16}$.

In type III, we can classify P's and Y's according to the length of their boundary lying on the boundary of the 6-6-6 (resp. 3-3-12) triangle. In the 6-6-6 triangle we obtain:

(1) length-1: 1 case, quadratic cone $(\mathbb{F}_2^0, \mathcal{O}(1))$ only (corresponding to A_0).
(2) length-2: 3 cases, including $(\mathbb{P}^2, \mathcal{O}(2))$ (corresponding to A_1^e, A_1^c, A_2').

(3) length-3: 2 cases (corresponding to A_2, A'_3).
(4) length-4: 4 cases (corresponding to A^e_3, A^c_3, D_4, A'_4).
(5) length-5: 3 cases (corresponding to A_4, D_5, A'_5).
(6) length-6: 4 cases (corresponding to A^e_5, D_6, A'_6, E_6).
(7) length-7: 3 cases (corresponding to D_7, A'_7, E_7).
(8) length-8: 3 cases (corresponding to D_8, E_8, A''_9).
(9) length-9: 1 case (corresponding to D'_{10}).

In the 3-3-12 triangle, in addition to most of those listed above, we also obtain:

(6) length-6: 1 case (corresponding to A^c_5).
(7) length-7: 1 case (corresponding to A_6).
(8) length-8: 3 cases (corresponding to A^e_7, A^c_7, A'_8).
(9) length-9: 3 cases (corresponding to A_8, D_9, A'_9).
(10) length-10: 4 cases (corresponding to A^e_9, A^c_9, D_{10}, A'_{10}).
(11) length-11: 3 cases (corresponding to A_{10}, D_{11}, A'_{11}).
(12) length-12: 3 cases (corresponding to A^e_{11}, D_{12}, A'_{12}).
(13) length-13: 2 cases (corresponding to D_{13}, A'_{13}).
(14) length-14: 2 cases (corresponding to D_{14}, A''_{15}).
(15) length-15: 1 case (corresponding to D'_{16}).

We obtain 47 cases in total and the polytopes P have 3, 4, or 5 sides, when P straddles 1, 2, or 3 sides of the big triangle, respectively (and the corresponding diagrams are of types $A^e_n/A^c_n/A_n$, $D_n/A'_n/E_n$, or D'_n/A''_n respectively).

Example 4.25. Let $Y = (\mathbb{F}^0_2, \mathcal{O}(1))$. Then Δ_1 is a line through the vertex of the quadratic cone, and Δ_2, B are two conic sections not passing through the vertex. Then $(Y, \Delta_1 + \Delta_2 + \frac{1}{2}B)$ is lc, and the only case when it is not lc for the coefficient $\frac{1}{2} + \epsilon$ is when B is tangent to Δ_2.

Example 4.26. Let $Y = (\mathbb{P}^2, \mathcal{O}(2))$. Then Δ_1, Δ_2 are two lines. Then again $(Y, \Delta_1 + \Delta_2 + \frac{1}{2}B)$ is lc, and the only case when it is not lc for the coefficient $\frac{1}{2} + \epsilon$ is when B is a conic in $\mathbb{P}^2$ which is either a double line or a conic tangent to either of Δ_i's or to both of them.

Lemma 4.27. *Suppose that a polytope P is contained in a strip of height 2. Write $B = B_1 + 2B_2$, where $2B_2$ is the double part. Moreover, if Y has one or two structures of generically $\mathbb{P}^1$-fibrations over $\mathbb{P}^1$ (e.g. $Y = \mathbb{F}_n$ or $\mathbb{P}^1 \times \mathbb{P}^1$), let $2B'_2$ be the part of $2B_2$ not contained in the fibers. Then:*

(1) $(X, \Delta + \frac{1}{2}B)$ is lc.
(2) $(X, (\frac{1}{2} + \epsilon)B_1)$ is lc.
(3) $B'_2 \cap B_1 = \emptyset$.

Proof. Either P has two parallel sides at lattice distance 2, or one boundary of the height-2 strip intersects P at the point $(2, 2, 2)$. In the first case, Y has a structure of a generically $\mathbb{P}^1$-fibration $|f|$, and $B.f = 2$. In the second case, there is such a structure after a blowup at the point corresponding to the vertex $(2, 2, 2)$. Thus, there is again a pencil of curves $|f|$ on Y such that $B.f = 2$.

Let $P \in Y \setminus \Delta$. Let F be a curve from the above pencil $|f|$ passing through P. Let the coefficient of F in B be $0 \le a \le 2$. The pair $(F, \frac{1}{2}(B - aF)|_F$ is lc, since $F.\frac{1}{2}(B - aF) = 2$. By inversion of adjunction, this implies that the pair $(Y, \frac{1}{2}(B + (2-a)F))$ is lc in a neighborhood of P. Moreover, the pair $(Y, \frac{1}{2}B)$ is lc, which proves (1) outside of Δ. Near Δ, again, we can use adjunction to Δ.

Moreover, if $a \le 1$, i.e. $F \not\subset B_2$ then it follows that minimal discrepancy over P is > -1. In particular, $(X, \frac{1}{2}B)$ is klt at P unless there is a double component of B passing through P. This proves (2). Part (3) follows similarly. □

Remark 4.28. This lemma leaves out only 3 out of a total of 47 polytopes. All of them are 4-gons and correspond to subdiagrams of type E_n. The sides on the boundary have lengths $3+3$ (corresponding to E_6), $3+4$ (corresponding to E_7), and $4+4$ (corresponding to E_8).

Lemma 4.29. *In the remaining 3 cases in type III, the 4-gons with boundary lengths $3+3$, $3+4$, and $3+5$, the following holds:*

(1) $(X, \Delta + \frac{1}{2}B)$ is lc.

(2) $(X, (\frac{1}{2}+\epsilon)B_1)$ is lc.

Proof. Let P be the 4-gon with the boundary lengths $3+3$. The corresponding polarized toric variety is obtained from $(\mathbb{P}^2, \mathcal{O}(4))$ by making weighted blowups at the coordinate points $x = z = 0$ and $y = z = 0$, and then contracting the line $z = 0$.

The equation of B in this case is $x^2y^2 + zg(x, y, z)$. Considering this as an equation of a quartic on $\mathbb{P}^2$, this implies that the pair $(\mathbb{P}^2, \Delta + \frac{1}{2}B)$ is lc in a neighborhood of Δ and is maximally lc at the points $x = z = 0$, $y = z = 0$. If B is a reduced quartic then B can have at worst an $\widetilde{E}_7$-singularity or Du Val singularities, see [BG81]. Obviously, B does not contain a triple line. It follows that the pair $(\mathbb{P}^2, \Delta + \frac{1}{2}B)$ is lc at it is maximally lc at the points $x = z = 0$, $y = z = 0$. It follows that the same is true for the pair $(Y, \Delta + \frac{1}{2}B)$, after the toric blowups and contraction. The case of the 4-gon with the boundary lengths $3+4$ is done by very similar arguments.

Finally, the case of the $3+5$ 4-gon may be treated as follows. The corresponding polarized toric variety is obtained from $(\mathbb{P}(1,1,2), \mathcal{O}(6))$, with weighted coordinates (x, y, z) in weights $(1,1,2)$ respectively, by making a weighted blow-up at the point $y = z = 0$.

The equation of B in this case is given by $x^2z^2 + az^3 + yg(x, y, z)$, where $a \neq 0$ (otherwise the polygon would be 2+5, not 3+5) and g is a quintic. The intersection of this curve with $\Delta = \{y = 0\}$ in $\mathbb{P}(1,1,2)$ is two points, one of which is doubled, so the pair $(\mathbb{P}(1,1,2), \Delta + \frac{1}{2}B)$ is lc in a neighborhood of Δ and is maximally lc at the point $y = z = 0$.

If B is a reduced divisor, then it follows from [Wal99, Theorem 5.1][1] that the double cover $\{w^2 = x^2z^2 + az^3 + yg(x, y, z)\} \subset \mathbb{P}(1,1,2,3)$ is a rational elliptic surface and the singularities on B are no worse than A-D-E. Moreover, it is obvious that B cannot contain a triple component, as it intersects Δ in two distinct points. It follows that the pair $(\mathbb{P}(1,1,2), \Delta + \frac{1}{2}B)$ is lc as it is maximally lc at the points $x = z = 0$, $y = z = 0$. Thus, the same is true for the pair $(Y, \Delta + \frac{1}{2}B)$ after the toric blowup. □

Corollary 4.30. *Assume that that P is not the entire 6-6-6 or 3-3-12 triangle, ie. $Y \neq (\mathbb{P}^2, \mathcal{O}(6))$ and $Y \neq (\mathbb{P}(1,1,4), \mathcal{O}(12))$. Assume that B intersected with any $\Delta_j = \mathbb{P}^1$ corresponding to a face passing through $(2,2,2)$ does not contain*

[1]We note that [Wal99] was never published, due to a substantial overlap between its main results and those in an earlier paper of Degtyarev [Deg90], which was discovered late in the editing process. However, we find the broader treatment in [Wal99] more directly applicable to our setting, so we prefer to use that reference here.

double points, i.e. that $(Y, \Delta + (\frac{1}{2} + \epsilon)B)$ is lc in a neighborhood of Δ_j. Then $(Y, \Delta + (\frac{1}{2} + \epsilon)B)$ is lc everywhere.

With these reductions in place, our strategy is to make elementary operations, similar to those in Section 4.4, until the central fiber is a stable surface $\big(X, (\frac{1}{2}+\epsilon)B\big)$. At every step, we see that the surfaces do not go out of the class of the "umbrella" type surfaces claimed in Theorem 4.1.

Example 4.31. Let $X_0 = \mathbb{P}^2$ and B_0 be a double cubic C. Blow up C in the 3-fold. A $\mathbb{P}^1$-fibration over C is inserted, and the restriction of B to the new model is a 2-section on this $\mathbb{P}^1$-fibration, which is disjoint from the exeptional section. If this 2-section is not a double section, then the pair $(X_0, \frac{1}{2} + \epsilon)B_0$ is slc, and passing to the canonical model consists of blowing down the original $\mathbb{P}^2$ component, leaving a cone over C with a 2-section. If, however, the 2-section is a double section, then we simply repeat the procedure by blowing this section up, etc. The end result is always a cone over C with a 2-section.

4.6. **Only surfaces claimed in Theorem 4.1 occur as stable limits: end of the proof.** From the previous section, possibly after performing a base change, we may assume that the following condition holds:

Condition 4.32. $(\mathcal{X}, \mathcal{B}) \to S$ is a family of pairs with generic fiber isomorphic to a sextic in $\mathbb{P}^2$ and special fiber (X_0, B_0), satisfying:

(1) X_0 is a surface of umbrella type, corresponding to an elliptic/maximal parabolic subdiagram of $\Gamma_{\rm Vin}$ as in Theorem 4.1.
(2) $\mathcal{B}$ is a horizontal $\mathbb{Q}$-Cartier divisor.
(3) $2K_{\mathcal{X}} + \mathcal{B} \sim 0$.
(4) $(\mathcal{X}, \mathcal{B}) \to \Delta$ admits a semistable log resolution.
(5) The pair $(\mathcal{X}, \frac{1}{2}\mathcal{B} + X_0)$ is log canonical.
(6) The divisor $K_{\mathcal{X}} + (\frac{1}{2} + \epsilon)\mathcal{B}$ is relatively ample over S for some $\epsilon > 0$.

Our aim is to prove the following:

Proposition 4.33. *After a finite series of birational modifications, we may assume that $(\mathcal{X}, \mathcal{B}) \to \Delta$ is a family of* stable *pairs with central fiber of umbrella type, corresponding to an elliptic/maximal parabolic subdiagram as in Theorem 4.1.*

Proof. We begin the proof of this proposition by noting that most of the conditions needed for $(\mathcal{X}, \mathcal{B})$ to be a family of stable pairs follow immediately from Condition 4.32. Indeed, we only need to show that the pair $(X_0, (\frac{1}{2} + \epsilon)B_0)$ can be made lc. Moreover, by Condition 4.32(5) and adjunction, we know that the pair $(X_0, \frac{1}{2}B_0)$ is lc.

To prove the proposition, we will show that there exists a sequence of elementary modifications, which are birational and preserve Condition 4.32. These elementary modifications are applied to loci in X_0 where $(X_0, \frac{1}{2}B_0)$ is lc but $(X_0, (\frac{1}{2} + \epsilon)B_0)$ is not (i.e. $(X_0, \frac{1}{2}B_0)$ is maximally lc). It follows from Corollary 4.30 that such loci fall into two classes: curves of multiplicity two in B_0 and points where B_0 intersects a double curve in X_0 non-transversely.

The elementary modifications we will use are of the following two types:

(a) Our first elementary modification generalizes those discussed in Section 4.4. Suppose that B_0 contains an irreducible curve C of multiplicity two and let Y denote the component of X_0 containing C. Since C is a log canonical center for the pair $(\mathcal{X}, \frac{1}{2}\mathcal{B}+X_0)$, we can extract a divisor E with discrepancy -1 from C to obtain a new threefold $\hat{\mathcal{X}}$. Let $\hat{\mathcal{B}}$ denote the strict transform of $\mathcal{B}$, and let $\hat{X}_0$ denote the total transform of X_0. Note that $\hat{\mathcal{B}}$ is relatively nef away from Y. Examining the possibilities for Y given by Tables 2 and 3, we thus see that $\hat{\mathcal{B}}$ only fails to be relatively nef when $\mathcal{B} \cap Y$ contains a curve of multiplicity two with self-intersection $-\frac{1}{2}$ (which can occur for diagrams of types A_n', D_n' and A_n''). The argument given in Section 4.4.2 shows that we may flip such curves, after which $\hat{\mathcal{B}}$ becomes relatively nef. Conclude the elementary modification by contracting any curves C' in $\hat{X}_0$ with $\hat{\mathcal{B}}.C' = 0$.

(b) Suppose that B_0 intersects a double curve in X_0 non-transversely in a point P. Since P is a log canonical center for the pair $(\mathcal{X}, \frac{1}{2}\mathcal{B}+X_0)$, we can extract a divisor E with discrepancy -1 from P to obtain a new threefold $\hat{\mathcal{X}}$. Let $\hat{\mathcal{B}}$ denote the strict transform of $\mathcal{B}$. Examining the possibilities for the components of X_0 containing P, as given by Tables 2 and 3, we see that $\hat{\mathcal{B}}$ must be relatively nef. Conclude the elementary modification by contracting any curves C' in $\hat{X}_0$ with $\hat{\mathcal{B}}.C' = 0$.

These elementary modifications preserve Condition 4.32. Indeed, the only difficult part is showing that Condition 4.32(1) is preserved. In the case of an elementary modification of type (a), one may use an analogous argument to that in Section 4.4 to show that the resulting surface is still of umbrella type. To see that the resulting surface corresponds to an elliptic/parabolic subdiagram, note that the exceptional component of the blow-up is ruled over C. After contracting, this exceptional component corresponds to a subdiagram of type $A_n^e/A_n^c/A_n$ (if $\hat{\mathcal{B}}$ is nef and C is contracted), D_n (if $\hat{\mathcal{B}}$ is nef and C is not contracted), A_n' (if one curve is flipped and C is contracted), D_n' (if one curve is flipped and C is not contracted), or A_n'' (if two curves are flipped).

In the case of an elementary modification of type (b), the new component is introduced between two existing ones, so the resulting surface is again of umbrella type. Moreover, the new component has Picard rank 1, so must correspond to a subdiagram of type $A_n^e/A_n^c/A_n$.

To prove Proposition 4.33, we perform the following algorithm:

(1) If B_0 contains a curve C of multiplicity two, perform an elementary modification of type (a) along C.
(2) Repeat step (1) until there are no curves of multiplicity two remaining in B_0.
(3) If there is a point P in X_0 where B_0 intersects a double curve non-transversely, perform an elementary modification of type (b) at P.
(4) Return to step (1).

If this algorithm terminates, by Corollary 4.30 all loci in X_0 where $(X_0, \frac{1}{2}B_0)$ is lc but $(X_0, (\frac{1}{2}+\epsilon)B_0)$ is not will have been removed, so $(X_0, (\frac{1}{2}+\epsilon)B_0)$ must be lc. By the following lemma, this completes the proof. □

Lemma 4.34. *The algorithm above terminates after finitely many steps.*

Proof. Begin by noting that since $(\mathcal{X}, \frac{1}{2}\mathcal{B}+X_0)$ is lc (by Condition 4.32(5)), for suitable choices of $0 < \epsilon \ll \delta$ the pair $(\mathcal{X}, (\frac{1}{2}+\epsilon)\mathcal{B}+(1-\delta)X_0)$ is klt. Fix such ϵ and δ.

Let $a(E,\mathcal{X},(\frac{1}{2}+\epsilon)\mathcal{B}+(1-\delta)X_0)$ denote the discrepancy of the divisor E with respect to the pair $(\mathcal{X},(\frac{1}{2}+\epsilon)\mathcal{B}+(1-\delta)X_0)$. Suppose that we perform an elementary modification taking $(\mathcal{X},\mathcal{B})$ to $(\mathcal{X}',\mathcal{B}')$. If E is the new component arising from this operation, then we will show that

$$0 > a(E,\mathcal{X}',(\tfrac{1}{2}+\epsilon)\mathcal{B}'+(1-\delta)X_0') = a(E,\mathcal{X},(\tfrac{1}{2}+\epsilon)\mathcal{B}+(1-\delta)X_0)+2\epsilon,$$

i.e. the discrepancy of E increases by 2ϵ but remains negative; whilst if F is any other divisor,

$$a(F,\mathcal{X}',(\tfrac{1}{2}+\epsilon)\mathcal{B}'+(1-\delta)X_0') \geq a(F,\mathcal{X},(\tfrac{1}{2}+\epsilon)\mathcal{B}+(1-\delta)X_0),$$

i.e. the discrepancy of F does not decrease. As, by [KM98, Proposition 2.36], there are only finitely many components E with negative discrepancy, this can only occur finitely many times, so the algorithm must terminate.

Consider an elementary modification of either type. Let $f\colon \hat{\mathcal{X}} \to \mathcal{X}$ denote the extraction, let $\hat{\mathcal{B}}$ denote the strict transform of $\mathcal{B}$, let E denote the exceptional divisor, and let $f_*^{-1}X_0$ denote the strict transform of X_0. Since the discrepancy of E with respect to the pair $(\mathcal{X},\frac{1}{2}\mathcal{B}+X_0)$ is equal to -1 by construction, we obtain $K_{\hat{\mathcal{X}}}+\frac{1}{2}\hat{\mathcal{B}}+f_*^{-1}X_0 \equiv f^*(K_{\mathcal{X}}+\frac{1}{2}\mathcal{B}+X_0)-E$. Moreover, by Condition 4.32(4), we obtain $\hat{X}_0 \sim f^*(X_0) \sim f_*^{-1}X_0+E$. Putting these together gives $K_{\hat{\mathcal{X}}}+\frac{1}{2}\hat{\mathcal{B}} \equiv f^*(K_{\mathcal{X}}+\frac{1}{2}\mathcal{B}) \equiv 0$, and thus

$$K_{\hat{\mathcal{X}}}+(\tfrac{1}{2}+\epsilon)\hat{\mathcal{B}}+(1-\delta)\hat{X}_0 \equiv f^*(K_{\mathcal{X}}+(\tfrac{1}{2}+\epsilon)\mathcal{B}+(1-\delta)X_0)-2\epsilon E,$$

which gives

$$\begin{aligned} a(E,\mathcal{X},(\tfrac{1}{2}+\epsilon)\mathcal{B}+(1-\delta)X_0) &= -1+\delta-2\epsilon,\\ a(E,\hat{\mathcal{X}},(\tfrac{1}{2}+\epsilon)\hat{\mathcal{B}}+(1-\delta)\hat{X}_0) &= -1+\delta<0. \end{aligned}$$

and, by [KM98, Lemmas 2.27 and 2.30], for any other divisor F we have

$$a(F,\hat{\mathcal{X}},(\tfrac{1}{2}+\epsilon)\hat{\mathcal{B}}+(1-\delta)\hat{X}_0) \geq a(F,\mathcal{X},(\tfrac{1}{2}+\epsilon)\mathcal{B}+(1-\delta)X_0).$$

The next step in the elementary modification is to flip any curves $\hat{C}$ in $\hat{\mathcal{X}}_0$ with $\hat{\mathcal{B}}_0.\hat{C}<0$ (which only occur for certain elementary modifications of type (a)), then contract any curves with $\hat{\mathcal{B}}_0.\hat{C}=0$. As E is not contracted this process does not affect the discrepancy of E and, by [KM98, Lemma 3.38], the discrepancy of F does not decrease for all other divisors F. This shows that elementary modifications have the required properties. □

5. Maximally log canonical sextics in the plane and the Γ_{21} diagram

We now digress to discuss a closely related result, which classifies maximally lc plane sextics in terms of parabolic/hyperbolic subdiagrams of a graph Γ_{21}, which is closely related to Γ_{Vin}.

As before, we let $(\mathcal{X}^*,\mathcal{B}^*) \to S\setminus 0$ denote a one-parameter degeneration of plane sextics. By results of Hacking [Hac04, Theorem 2.6], after a finite surjective base change the family $(\mathcal{X}^*,\mathcal{B}^*) \to S\setminus 0$ may be completed (possibly non-uniquely) to a flat family of *semistable pairs* $(\mathcal{X},\mathcal{B}) \to S$ of degree 6, defined below, so that $K_{\mathcal{X}}$ and $\mathcal{B}$ are $\mathbb{Q}$-Cartier divisors with $2K_{\mathcal{X}}+\mathcal{B}\sim 0$.

Definition 5.1. [Hac04, 2.4] Let X be a surface and let B be an effective $\mathbb{Q}$-Cartier divisor on X. Then (X,B) is a *semistable pair of degree* 6 if

(1) the surface X is normal and log terminal,
(2) the pair $(X, \frac{1}{2}B)$ is lc,
(3) the divisor $2K_X + B$ is linearly equivalent to zero, and
(4) there is a deformation $(\mathcal{X}, \mathcal{B}) \to \Delta$ of the pair (X, B) so that the general fiber of $\mathcal{X}$ is isomorphic to $\mathbb{P}^2$ and the divisors $K_{\mathcal{X}}$ and $\mathcal{B}$ are $\mathbb{Q}$-Cartier.

Now, let (X, B) be a semistable pair of degree 6 and let $f\colon \hat{X} \to X$ be a minimal log resolution of (X, B). Define an effective $\mathbb{Q}$-divisor $\hat{B}$ to be the divisor on $\hat{X}$ with $\mathrm{Supp}(\hat{B}) = \mathrm{Supp}(f^*B)$ and $K_{\hat{X}} + \frac{1}{2}\hat{B} \equiv f^*(K_X + \frac{1}{2}B) \equiv 0$. Then (X, B) is said to be of

- *Type I* if the integral part $\lfloor \frac{1}{2}\hat{B} \rfloor$ of $\frac{1}{2}\hat{B}$ is empty,
- *Type II* if $\lfloor \frac{1}{2}\hat{B} \rfloor$ is nonempty and smooth (but not necessarily connected), and
- *Type III* if $\lfloor \frac{1}{2}\hat{B} \rfloor$ contains simple normal crossing singularities.

The rationale behind these names is as follows. Let $\mathcal{Y} \to \Delta$ denote the double cover of $\mathcal{X} \to \Delta$ ramified over $\mathcal{B}$. Then $\mathcal{Y} \to \Delta$ is a flat family of K3 surfaces of degree 2. Let $\hat{\mathcal{Y}} \to \Delta$ denote a semistable resolution of $\mathcal{Y} \to \Delta$ with special fiber $\hat{Y}$, then $\hat{Y}$ is a degenerate K3 surface of Type I (resp. II, III) if and only if the special fiber (X, B) of $(\mathcal{X}, \mathcal{B}) \to \Delta$ is of Type I (resp. II, III).

5.1. **Statement of the result.** The aim of this section is to classify semistable pairs (X, B) of degree 6, where we make the following simplifying assumptions:

Assumption 5.2. We assume that (X, B) is a semistable pair of degree 6 satisfying

(1) the surface $X \cong \mathbb{P}^2$, and
(2) the pair $(X, \frac{1}{2}B)$ is *maximally lc*, i.e. $(X, \frac{1}{2}B)$ is lc but $(X, (\frac{1}{2} + \epsilon)B)$ is not lc for any $\epsilon > 0$.

Remark 5.3. Part (1) of this assumption implies that K_X is Cartier so, by Definition 5.1, B is Cartier also. It is easy to see that part (2) of this assumption occurs if and only if the pair (X, B) is of Type II or Type III.

Our aim is to classify semistable pairs $(\mathbb{P}^2, B)$ satisfying these assumptions, up to the following notion of equivalence:

Definition 5.4. Let $(\mathbb{P}^2, B)$ and $(\mathbb{P}^2, B')$ be two semistable pairs of of degree 6, let $f\colon \hat{X} \to \mathbb{P}^2$ (resp. $f'\colon \hat{X}' \to \mathbb{P}^2$) be minimal log resolutions of the log canonical loci of the pairs $(\mathbb{P}^2, \frac{1}{2}B)$ (resp. $(\mathbb{P}^2, \frac{1}{2}B')$), and let $\hat{B}$ (resp. $\hat{B}'$) denote the strict transforms of B (resp. B') under these resolutions. Then we say that $(\mathbb{P}^2, B)$ and $(\mathbb{P}^2, B')$ are *equivalent* if:

- $\hat{X}$ and $\hat{X}'$ are isomorphic, and
- $\hat{B}$ and $\hat{B}'$ are linearly equivalent divisors on $\hat{X} \cong \hat{X}'$.

Our classification will be given in terms of certain subdiagrams of the diagram Γ_{21}, which is given in Figure 7. For later use we have labeled the 21 vertices of Γ_{21} with letters s_i and t_j. This labeling divides the vertices into two types: 3 *corner vertices*, which are labeled s_i in Figure 7, and 18 *outer vertices*, which are labeled t_j.

Now we are ready to state the main theorem of this section.

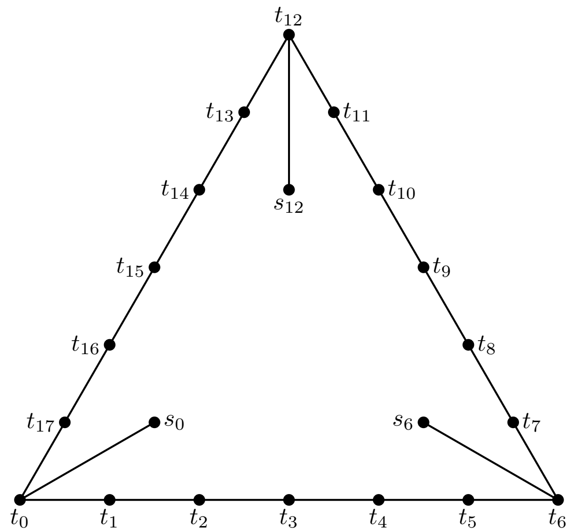


FIGURE 7. The diagram Γ_{21}

Theorem 5.5. *Let $(\mathbb{P}^2, B)$ be a semistable pair of degree 6 satisfying Assumption 5.2. There is a bijective correspondence between equivalence classes of such pairs $(\mathbb{P}^2, B)$ (under the equivalence given by Definition 5.4) and subdiagrams Γ of Γ_{21}, up to the action of S_3, such that every connected component of Γ is either parabolic or hyperbolic. Furthermore,*

- *the pair $(\mathbb{P}^2, B)$ is of Type II (resp. Type III) if and only if every connected component of the corresponding subdiagram is parabolic (resp. hyperbolic), and*
- *degeneration of pairs corresponds to inclusion of subdiagrams, with larger subdiagrams being more degenerate.*

5.1.1. *Interpretation.* This result may be interpreted as follows. Define a sextic by $\sum u_i = 0$, where

$u_c = (-xyz + s_0x^3 + s_6y^3 + s_{12}z^3)^2$

$u_0 = t_0x^6$	$u_1 = t_1x^4(-xy + s_{12}z^2)$	$u_2 = t_2x^2(-xy + s_{12}z^2)^2$
$u_3 = t_3(-xy + s_{12}z^2)^3$	$u_4 = t_4y^2(-xy + s_{12}z^2)^2$	$u_5 = t_5y^4(-xy + s_{12}z^2)$
$u_6 = t_6y^6$	$u_7 = t_7y^4(-yz + s_0x^2)$	$u_8 = t_8y^2(-yz + s_0x^2)^2$
$u_9 = t_9(-yz + s_0x^2)^3$	$u_{10} = t_{10}z^2(-yz + s_0x^2)^2$	$u_{11} = t_{11}z^4(-yz + s_0x^2)$
$u_{12} = t_{12}z^6$	$u_{13} = t_{13}z^4(-xz + s_6y^2)$	$u_{14} = t_{14}z^2(-xz + s_6y^2)^2$
$u_{15} = t_{15}(-xz + s_6y^2)^3$	$u_{16} = t_{16}x^2(-xz + s_6y^2)^2$	$u_{17} = t_{17}x^4(-xz + s_6y^2)$

The coefficients s_i, t_j appearing in this expression can be associated to vertices of the Γ_{21} diagram as shown in Figure 7. Thus, given a parabolic/hyperbolic subdiagram, one may define a sextic equation by setting the s_i, t_j which correspond to vertices in that subdiagram to zero. The resulting sextic in $\mathbb{P}^2$ is a generic member of the equivalence class corresponding to that parabolic/hyperbolic subdiagram.

Remark 5.6. The rule for generating the above expressions for u_i is as follows. To each of the 18 vertices on the boundary of Figure 7, associate the corresponding monomial $x^\ell y^m z^n$ from Figure 3b, and add in an additional monomial $x^2y^2z^2$

corresponding to the centre. Multiply the monomials on the boundary by the corresponding t_i (formally, $t_c = 1$), to give 19 monomial expressions

$$u_c = (xyz)^2 \quad u_0 = t_0 x^6 \quad u_1 = t_1 x^4(xy) \quad u_2 = t_2 x^2(xy)^2 \quad u_3 = t_3(xy)^3 \quad \text{etc.}$$

Setting $t_i = t^{h_i}$ for some heights $h_i \in \mathbb{Z}$ gives all the possible 1-parameter limits in the first toric family of Section 4.2 (c.f. Section 4.3). This is essentially the meaning of the toric construction.

Now "deform" these formulas by making the substitutions

$$xy \mapsto -xy + s_{12}z^2, \quad yz \mapsto -yz + s_0x^2, \quad zx \mapsto -xz + s_6y^2,$$
$$xyz \mapsto -xyz + s_0x^3 + s_6y^3 + s_{12}z^3.$$

Adding the minus signs is not strictly necessary at this stage, but will make the formulas in the next section somewhat easier.

5.2. **Beginning the proof of Theorem 5.5.** The remainder of this section will be devoted to proving Theorem 5.5. We will make repeated use of the fact that any semistable pair $(\mathbb{P}^2, B)$ of degree 6 satisfying Assumption 5.2 determines a *sextic double plane* Y, given by the double cover of $\mathbb{P}^2$ branched along the divisor B. Note that Y may be non-normal (if B is not reduced), but that the assumptions on $(\mathbb{P}^2, B)$ imply that it must have *Gorenstein semi log canonical* singularities.

Definition 5.7. A surface Y is said to have *semi log canonical* (*slc*) singularities if the pair $(Y, 0)$ is slc (see Definition 3.2). Y has *Gorenstein semi log canonical* singularities if, in addition, K_Y is Cartier.

A coarse classification of singularities of this type was given by Kollár and Shepherd-Barron in [KSB88, Theorem 4.21]; their result states that a Gorenstein surface singularity is slc if and only if it is isomorphic to one of

- a rational double point (RDP) $0 \in \{z^2 = f(x, y)\} \subset \mathbb{C}^3$, where the branch curve $\{f(x, y) = 0\} \subset \mathbb{C}^2$ has an A-D-E singularity at $0 \in \mathbb{C}^2$;
- a double normal crossing point $0 \in \{xy = 0\} \subset \mathbb{C}^3$;
- a pinch point $0 \in \{x^2 = zy^2\} \subset \mathbb{C}^3$;
- a simple elliptic singularity;
- a cusp;
- a degenerate cusp.

In this classification, simple elliptic, cusp and degenerate cusp singularities are defined by the form of their minimal semi-resolutions (see [KSB88, Section 4] for the definition of a minimal semi-resolution).

Definition 5.8. [KSB88, 4.20] A Gorenstein surface singularity is called:

- Simple elliptic if it is normal and the exceptional divisor of the minimal resolution is a smooth elliptic curve.
- A cusp if it is normal and the exceptional divisor of the minimal resolution is a cycle of smooth rational curves or a rational nodal curve.
- A degenerate cusp if it is not normal and the exceptional divisor of the minimal semi-resolution is a cycle of smooth rational curves or a rational nodal curve.

We will use this classification as the starting point for our analysis. However, as any RDP arising in a sextic double plane comes from an A-D-E singularity in the branch divisor B, which is *not* a maximally log canonical singularity of the pair

$(\mathbb{P}^2, \frac{1}{2}B)$, by Assumption 5.2(2) we may assume that Y is a sextic double plane that contains at least one Gorenstein slc singularity that is not an RDP.

To further simplify matters, we also note that the sextic double plane Y can be realized as a hypersurface $Y = \{z^2 = f_6(x_i)\} \subset \mathbb{P}(1,1,1,3)$, where $\{f_6(x_i) = 0\}$ is the equation of the divisor B in $\mathbb{P}^2$. From this it is easy to see that any singularity arising in Y must have embedding dimension 3 (i.e. be a hypersurface singularity) and multiplicity 2.

5.3. **Simple elliptic singularities.** We begin by assuming that Y is normal and has at worst simple elliptic singularities (i.e. Y does not contain any cusps). As these singularities are resolved by a single smooth elliptic exceptional curve, in this case the pair $(\mathbb{P}^2, B)$ is always of Type II.

Simple elliptic singularities of embedding dimension 3 have been classified by Saito [Sai74, Satz 1.9]. He finds three cases, distinguished by the self-intersection number of the exceptional elliptic curve in the minimal resolution. Only two of these cases, $\tilde{E}_7$ (where B has a *quadruple point*) and $\tilde{E}_8$ (where B has a *consecutive triple point*), have multiplicity 2; these correspond to exceptional elliptic curves with self-intersection numbers -2 and -1 respectively. The local equations of these singularities are

$$\tilde{E}_7\colon \quad 0 \in \{z^2 = xy(y-x)(y-\lambda x)\} \subset \mathbb{C}^3, \qquad \lambda \in \mathbb{C} - \{0,1\},$$
$$\tilde{E}_8\colon \quad 0 \in \{z^2 = y(y-x^2)(y-\lambda x^2)\} \subset \mathbb{C}^3, \qquad \lambda \in \mathbb{C} - \{0,1\}.$$

It is easily seen that both of these singularities can occur in sextic double planes. Furthermore, by [Wal99, Theorem 3.2], any sextic double plane having at worst simple elliptic singularities must have either one $\tilde{E}_7$, one $\tilde{E}_8$ or two $\tilde{E}_8$'s. By explicitly computing resolutions, it may be seen that each of these three cases forms a single equivalence class under Definition 5.4.

Each of these simple elliptic singularities already corresponds to a parabolic diagram, specifically, the extended Dynkin diagrams $\tilde{E}_7$ and $\tilde{E}_8$. These parabolic diagrams embed into our standard diagram as shown in Figure 8: $\tilde{E}_7$ embeds as Figure 8a (which, for consistency with the rest of the paper, we call $T_{4,4}$) and $\tilde{E}_8$ embeds as Figure 8b (which we call $T_{3,6}$).

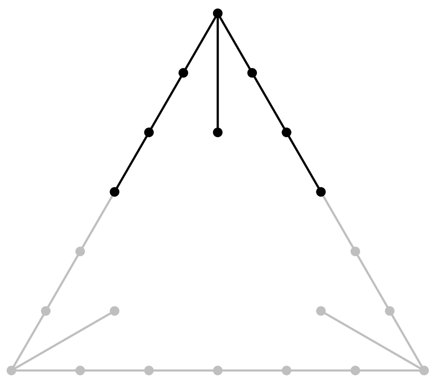
Note that it is also possible to simultaneously embed two diagrams of type $T_{3,6}$ into our standard diagram without them intersecting; this corresponds to the case when Y has two singularities of type $\tilde{E}_8$. By [Wal99, Theorem 3.2], this can only happen when the divisor B consists of three conics (one of which may degenerate to a pair of lines) meeting in two consecutive triple points.

5.4. **Cusp singularities.** Next we assume that Y is normal but contains at least one cusp singularity. As the exceptional divisors arising from resolving these singularities always contain simple normal crossing singularities, in this case the pair $(\mathbb{P}^2, B)$ is always of Type III.

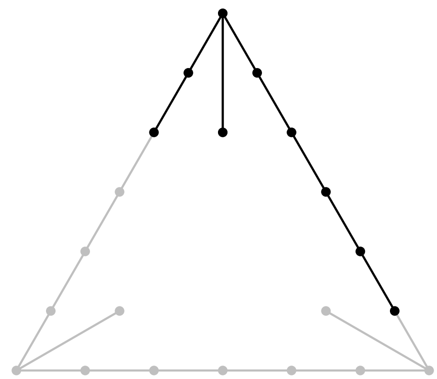
Cusp singularities of embedding dimension 3 generalize the simple elliptic singularities studied in Section 5.3. They have been studied by Arnold [Arn76], who found that they have the general form

$$T_{p,q,r}\colon \quad 0 \in \{x^p + y^q + z^r + \lambda xyz = 0\} \subset \mathbb{C}^3, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1, \quad \lambda \in \mathbb{C} - \{0\},$$

where the integers p, q, r are determined by the form of the exceptional locus in the minimal resolution. Of these, the cusp singularities with multiplicity 2 correspond



(A) The $T_{4,4}$ subdiagram



(B) The $T_{3,6}$ subdiagram

FIGURE 8. Parabolic subdiagrams of Γ_{21} corresponding to pairs $(\mathbb{P}^2, B)$ where B contains a quadruple point or consecutive triple point

.

to those $T_{p,q,r}$ with $p = 2$. Such singularities fall into two classes: $T_{2,3,r}$ with $r \geq 7$ (where B has a *higher triple point*) and $T_{2,q,r}$ with $q \geq 4$ and $r \geq 5$ (where B has a *higher quadruple point*). The forms of the exceptional loci appearing in the minimal resolutions corresponding to these cases have been calculated by Laufer in [Lau77, Section V]: in both cases the exceptional locus is a cycle of rational curves $E = \sum_i E_i$ with components satisfying $E_i^2 \leq -2$, that has $E^2 = -1$ (in the $T_{2,3,r}$ with $r \geq 7$ case) or $E^2 = -2$ (in the $T_{2,q,r}$ with $q \geq 4$ and $r \geq 5$ case).

Normal sextic double planes with cusp singularities are discussed in [Wal99]. In particular, [Wal99, Theorem 3.3] implies that if Y is normal and has a cusp singularity, then any other singularities it has must all be RDP's.

5.4.1. *Higher triple points.* Assume first that the cusp singularity has type $T_{2,3,r}$, with $r \geq 7$. Such singularities are discussed in [Wal99, Section 5]. We find:

Proposition 5.9. *Let Y be a normal sextic double plane determined by a semistable pair $(\mathbb{P}^2, B)$ satisfying Assumption 5.2, and suppose that Y has a cusp singularity of type $T_{2,3,r}$ with $r \geq 7$. Then B is a reduced curve in $\mathbb{P}^2$ and, up to the notion of equivalence from Definition 5.4, either:*

- *Y has a singularity of type $T_{2,3,r}$, with $7 \leq r \leq 15$, and B is irreducible,*
- *Y has a singularity of type $T_{2,3,8}$ and B contains a line through the higher triple point, or*
- *Y has a singularity of type $T_{2,3,14}$ and B contains a conic through the higher triple point.*

Moreover, all cases are realized.

Proof. The fact that B is reduced is immediate from normality of Y. By the argument in [Wal99, Section 5], B has the general form

$$\{x^3z^3 + x^2z^2a(y,z) + xzb(y,z) + c(y,z) = 0\} \subset \mathbb{P}^2,$$

where a, b and c are homogeneous functions in y and z of degrees 2, 4 and 6 respectively. After performing a blow-up and a change of coordinates, this function defines a rational elliptic surface. The form of B and the type of singularity occurring in it can then be read off from the form of this rational elliptic surface. Furthermore this construction can be reversed, so every rational elliptic surface gives rise to a sextic with a higher triple point.

Suppose first that B does not contain a line through the higher triple point. Then the corresponding rational elliptic surface does not contain any multiple fibers and, by [Wal99, Table 5], we see that a singularity of type $T_{2,3,r}$ gives rise to a singular fiber of type I_{r-6} (here we remark that, in Wall's notation, a singularity of type $T_{2,3,r}$ is denoted $E_{2,r-6}$).

Given this, Persson's classification of rational elliptic surfaces [Per90] shows that rational elliptic surfaces with a singular fiber of type I_k exist for exactly $1 \leq k \leq 9$, giving the required bounds on r. Furthermore, it follows from Persson's classification that, in all cases except $k = 8$, up to equivalence the curve B may be taken to be irreducible (corresponding to a rational elliptic surface with one singular fiber of type I_k and all other fibers of type I_1). When $k = 8$ (so $r = 14$) there are two distinct rational elliptic surfaces in Persson's classification with one I_8 fiber and all other fibers of type I_1; these correspond to cases where B is irreducible and where B splits as the union of a conic and a quartic. By explicitly computing resolutions, it is easy to see that these two cases are not equivalent.

Finally, suppose that B has a line L as a component. Write $B = B' + L$. Then c is divisible by z in the above expression. If b is not also divisible by z then, by [Wal99, Theorem 5.4], B' and L meet away from the higher triple point in an additional A_1 singularity and, by computing a resolution, we see that this case is equivalent to the irreducible case above. This leaves the case where b is divisible by z, in which case an application of [Wal99, Theorem 5.5] gives that Y has a singularity of type $T_{2,3,8}$. Another explicit resolution shows that this case is not equivalent to any of the others. □

5.4.2. *Higher quadruple points.* Assume next that the cusp singularity has type $T_{2,q,r}$, with $q \geq 4$ and $r \geq 5$. Such singularities are discussed in [Wal99, Section 6]. In analogy with the previous section we find:

Proposition 5.10. *Let Y be a sextic double plane determined by a semistable pair $(\mathbb{P}^2, B)$ satisfying Assumption 5.2, and suppose that Y has a cusp singularity of type $T_{2,q,r}$ with $q \geq 4$ and $r \geq 5$. Then, up to the notion of equivalence from Definition 5.4, either:*

- *Y has a singularity of type $T_{2,q,r}$, for $q+r \leq 18$ and $(q,r) \neq (4,14)$, $(6,12)$, $(8,10)$, and B is an irreducible curve.*
- *Y has a singularity of type $T_{2,q,r}$, for $(q,r) = (4,14)$, $(6,12)$ or $(8,10)$, and B is a union of two cubics, both of which pass through the quadruple point.*
- *Y has a singularity of type $T_{2,14,r}$, for $4 \leq r \leq 5$, and B is the union of a smooth conic and a quartic, both of which pass through the quadruple point.*
- *Y has a singularity of type $T_{2,8,r}$, for $4 \leq r \leq 11$, and B is the union of a line and a quintic, both of which pass through the quadruple point.*
- *Y has a singularity of type $T_{2,8,8}$ and B is the union of two lines and a quartic, all of which pass through the quadruple point.*

Moreover, all cases are realized.

Proof. By the results of [Wal99, Section 6], B can be written in the general form

$$\{F(x,y,z) = x^2a(y,z) + xb(y,z) + c(y,z) = 0\} \subset \mathbb{P}^2,$$

where a, b and c are functions of y and z of degrees 4, 5 and 6 respectively.

Given this, the possible higher quadruple points in B are given by partitions of 4 and 10, corresponding to factorization types of a and $\delta = b^2 - 4ac$ respectively (i.e. the numbers in each partition correspond to the multiplicities of the linear factors appearing in the corresponding factorization), that have been marked to show which factors of a also divide δ and, for any that do, which also divide F. Note that not all higher quadruple points are cusp singularities of type $T_{2,q,r}$, so we will also have to fulfill certain compatibility conditions:

(1) No term appearing in the partition of 4 may be greater than 2.
(2) Any factor of a that also divides δ must do so with equal or greater multiplicity.

Furthermore we note that, by [Wal99, Lemma 6.1], if B does not contain a line through the quadruple point, then B is reducible if and only if all values appearing in the partition of 10 are even.

We begin with the simplest partition of 4, given by $[1, 1, 1, 1]$. In this case, Y is a sextic double plane with a singularity of type $\tilde{E}_7$, which is a simple elliptic singularity and was treated in Section 5.3.

The next partition of 4 that we will consider is $[2, 1, 1]$. In this case we have several possibilities to consider:

- If the double factor of a is not a factor of δ, then Y is a sextic double plane with a singularity of type $T_{2,4,5}$. Resolving we see that, up to equivalence, B is an irreducible sextic, corresponding to the partition $[1^{10}]$ of 10.
- Suppose next that the double factor of a is a factor of δ, but that this factor is not also a factor of F. Then in order to satisfy (2) the corresponding partition of 10 must contain at least one value $2 \leq k \leq 10$, which corresponds to the double factor of a. Partitions of this type give rise to singularities of type $T_{2,4,k+4}$. Resolving, we see that each case $2 \leq k \leq 9$ gives rise to a single equivalence class, the generic member of which is a smooth sextic corresponding to the partition $[k, 1^{10-k}]$ of 10.

  The case $k = 10$ remains. In this case [Wal99, Lemma 6.1] shows that B must be reducible. Moreover, as we have assumed that the common factor of a and δ is not also a factor of F, we see that B cannot contain a line through the quadruple point. Resolving the $T_{2,4,14}$ singularity we thus see that, up to equivalence, there are two distinct possibilities: B is a union of two irreducible curves of degrees either $(3, 3)$ or $(2, 4)$, and both components pass through the quadruple point. By the proof of [Wal99, Lemma 6.1], both cases may occur.
- Finally, suppose that the double factor of a is a factor of both δ and F. Then B splits as the union of a line and a quintic, both of which pass through the singularity. In this case the quadruple point has type $T_{2,4,6}$ or $T_{2,4,8}$, depending upon the degree of tangency between the line and the quintic. Resolving, we find that the first case is equivalent to an irreducible sextic, which has already been covered, but the second is distinct.

The last partition of 4 that we need to consider is $[2, 2]$. In this case again there are several possibilities:

- If no factor of a is a factor of δ, then Y is a sextic double plane with a singularity of type $T_{2,5,5}$. Resolving we see that, up to equivalence, B is an irreducible sextic, corresponding to the partition $[1^{10}]$ of 10.
- If one factor of a is also a factor of δ, but not a factor of F, then the corresponding partition of 10 must contain at least one value $2 \leq k \leq 10$, which corresponds to the double factor of a. Such partitions give rise to singularities of type $T_{2,5,k+4}$. Resolving, we see that each case $2 \leq k \leq 9$ gives rise to a single equivalence class, the generic member of which is a smooth sextic corresponding to the partition $[k, 1^{10-k}]$ of 10.

  In the case $k = 10$, [Wal99, Lemma 6.1] shows that B must be reducible. Resolving the $T_{2,5,15}$ singularity, we see that B must be equivalent to a union of two irreducible curves of degrees $(2, 4)$. In particular, a singularity of type $T_{2,5,14}$ may only occur if B contains a conic.
- If both factors of a are also factors of δ, but neither is a factor of F, then in order to satisfy (2) the corresponding partition of 10 must contain two values $m, n \geq 2$, with $m + n \leq 10$, which correspond to the two common factors with a. Such partitions give rise to singularities of type $T_{2,m+4,n+4}$. Resolving, we see that each pair (m, n), except $(m, n) = (2, 8)$ and $(4, 6)$, gives rise to a single equivalence class, the generic member of which is a smooth sextic corresponding to the partition $[m, n, 1^{10-m-n}]$ of 10.

  In the cases where $(m, n) = (2, 8)$ or $(4, 6)$, by [Wal99, Lemma 6.1] we see that B must be reducible. Resolving the quadruple point singularity, we find that B must be a union of two irreducible curves of degrees $(3, 3)$, both of which pass through the quadruple point.
- a, δ and F all have a common factor if and only if B contains a line through the quadruple point. There are three subcases:
  - Suppose that precisely one factor of a is also a factor of δ and F. Then B splits as the union of a line and a quintic, which gives rise to a singularity of type $T_{2,5,6}$ or $T_{2,5,8}$, depending upon the degree of tangency between the line and the quintic. The first case is equivalent to an irreducible sextic, which has already been covered, but the second is distinct.
  - Next suppose that both factors of a are also factors of δ and one of these is also a factor of F. Then in order to satisfy (2) the corresponding partition of 10 must contain two values $m, n \geq 2$, with $m + n \leq 10$, which correspond to the common factors with a, and one of these (m say) also corresponds to a common factor with F. In this case B splits as the union of a line and a quintic, which gives rise to a singularity of type $T_{2,6,n+4}$ (if $m = 2$) or $T_{2,8,n+4}$ (if $m = 3$). In the case $T_{2,6,n+4}$ we find that B is equivalent to an irreducible sextic, so this case may be ignored. However, the case $T_{2,8,n+4}$ is new, and may occur for any $2 \leq n \leq 7$. This also shows that a singularity of type $T_{2,8,11}$ may only occur if B contains a line.
  - If both factors of a are also factors of both δ and F, then B consists of two lines through the quadruple point and a quartic. In this case B contains a singularity of type $T_{2,6,6}$, $T_{2,6,8}$ or $T_{2,8,8}$, depending upon the degrees of tangency between the lines and the branches of the quartic. The first two of these cases are equivalent to an irreducible

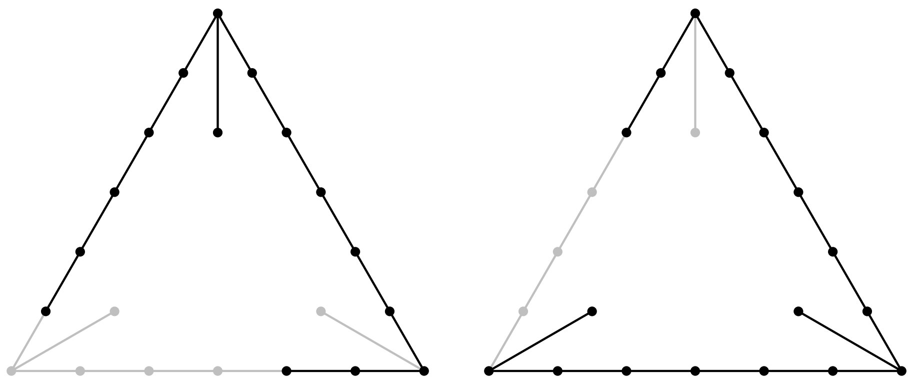
sextic and the union of a line and a quintic respectively, but the third is new.

As we have considered all possible partitions subject to the compatibility conditions (1) and (2), this exhausts the possibilities for B. □

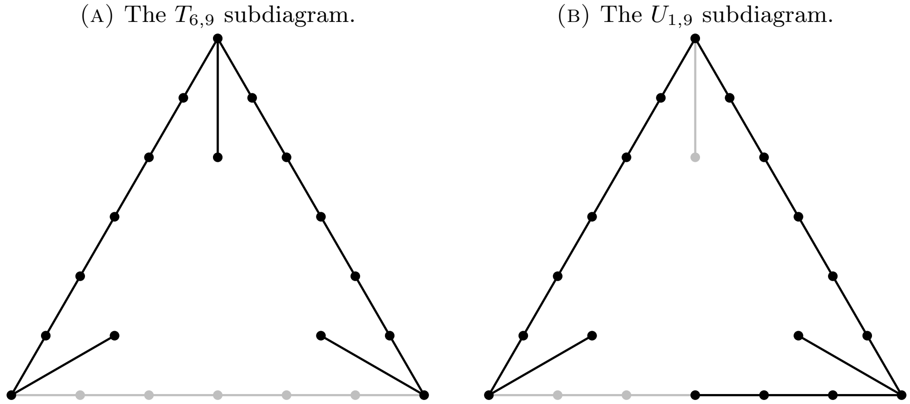
5.4.3. *Corresponding hyperbolic subdiagrams.* It remains to see how equivalence classes of semistable pairs $(\mathbb{P}^2, B)$ of degree 6 satisfying Assumption 5.2 that determine normal sextic double planes Y with cusp singularities correspond to hyperbolic subdiagrams of Γ_{21}. Note first that a cusp singularity of type $T_{2,q,r}$ already corresponds to a hyperbolic diagram (also denoted $T_{2,q,r}$), which embeds into our standard diagram. However, the type of embedding depends upon the form of the divisor B in a generic member of the equivalence class. There are 4 cases:

(1) The divisor B does not contain any components that are lines or smooth conics. By Propositions 5.9 and 5.10, in this case Y contains a cusp singularity of type $T_{2,q,r}$ with either $q = 3$, $r \geq 7$ or $q \geq 4$, $r \geq 5$, and in both cases $q + r \leq 18$. The hyperbolic diagram $T_{2,q,r}$ embeds into Γ_{21} in a way that incorporates precisely one of the corner vertices. Subdiagrams of this type will be referred to as *subdiagrams of type* $T_{q,r}$. Figure 9a illustrates the $T_{6,9}$ subdiagram, corresponding to a pair $(\mathbb{P}^2, B)$, with B irreducible, that determines a sextic double plane Y with a singularity of type $T_{2,6,9}$.
(2) The divisor B contains precisely one line. By Propositions 5.9 and 5.10, in this case Y contains a cusp singularity of type $T_{2,8,r}$ with $3 \leq r \leq 11$. The hyperbolic diagram $T_{2,8,r}$ embeds into Γ_{21} in a way that incorporates precisely two of the corner vertices. For consistency with later notation and to avoid confusion with case (1), subdiagrams of this type will be referred to as *subdiagrams of type* $U_{1,r}$. Figure 9b illustrates a diagram of type $U_{1,9}$, corresponding to a pair $(\mathbb{P}^2, B)$, where B contains precisely one line, that determines a sextic double plane Y with a singularity of type $T_{2,8,9}$.
(3) The divisor B contains two lines. By Proposition 5.10, this can only occur if Y contains a cusp singularity of type $T_{2,8,8}$. In this case the hyperbolic diagram $T_{2,8,8}$ embeds into Γ_{21} in a way that incorporates all three of the corner vertices (see Figure 9c). For consistency with later notation and to avoid confusion with cases (1) and (2), this subdiagram will be referred to as the *subdiagram of type* $W_{1,1}$.
(4) The divisor B contains a smooth conic. By Propositions 5.9 and 5.10, in this case Y contains a cusp singularity of type $T_{2,14,r}$ with $3 \leq r \leq 5$. The hyperbolic diagram $T_{2,14,r}$ embeds into Γ_{21} in a way that incorporates precisely two of the corner vertices. For consistency with later notation and to avoid confusion with the other cases, subdiagrams of this type will be referred to as *subdiagrams of type* $V_{1,r}$. Figure 9d illustrates a diagram of type $V_{1,4}$, corresponding to a pair $(\mathbb{P}^2, B)$, where B contains a smooth conic, that determines a sextic double plane Y with a singularity of type $T_{2,14,4}$.

5.5. **Non-normal singularities.** This exhausts the possibilities where the sextic double plane Y determined by $(\mathbb{P}^2, B)$ is normal. In the remaining sections we turn our attention to non-normal sextic double planes, i.e. those containing double normal crossing points, pinch points and degenerate cusps.



(A) The $T_{6,9}$ subdiagram. (B) The $U_{1,9}$ subdiagram.



(C) The $W_{1,1}$ subdiagram. (D) The $V_{1,4}$ subdiagram.

FIGURE 9. Hyperbolic subdiagrams of Γ_{21} corresponding to pairs $(\mathbb{P}^2, B)$ where B contains a higher triple point or higher quadruple point

In terms of the divisor B, such singularities occur when B contains a double component. We divide our analysis between the cases where this double component is a line, a pair of lines, a smooth conic or a (possibly singular) cubic.

Before we begin, however, we briefly discuss degenerate cusps. Degenerate cusps of embedding dimension 3 and multiplicity 2 have been classified by Shepherd-Barron in [SB83, Lemma 1.3]. In the proof of this lemma he shows that there are two possibilities, with local equations $0 \in \{z^2 = x^2y^2\} \subset \mathbb{C}^3$ and $0 \in \{z^2 = y^2(y^n + x^2)\} \subset \mathbb{C}^3$ (where $n \geq 1$).

Thus, we see that degenerate cusps can arise as follows. Either

- two double components of B meet at a node, giving a degenerate cusp of type $0 \in \{z^2 = x^2y^2\} \subset \mathbb{C}^3$ in Y, or
- a double component and a single component of B meet with local equation $0 \in \{0 = y^2(y^n + x^2)\} \subset \mathbb{C}^2$, where $n \geq 1$. This gives rise to a degenerate cusp of type $0 \in \{z^2 = y^2(y^n+x^2)\} \subset \mathbb{C}^3$ in Y. Given the obvious similarity to an A-D-E singularity of type D_{n+2}, we denote such a singularity by $\mathbb{D}_{n+2}$.

From this classification, it is easy to see that $(\mathbb{P}^2, B)$ is of Type II if and only if both the double and single components of B are smooth and meet transversely (giving only pinch point singularities); in all other cases $(\mathbb{P}^2, B)$ is of Type III.

5.5.1. *A double line.* We now turn our attention to the case where B contains precisely one double line L. In this setting B is the sum of $2L$ and a reduced quartic curve Q. The singularities present in the sextic double plane Y will depend upon the singularities present in the quartic Q and their position with respect to L. Singular quartics have been studied by Bruce and Giblin [BG81], we will rely heavily upon their results here.

If Q contains a singularity that is worse than A-D-E, then Bruce and Giblin [BG81, Section 1] show that it must consist of 4 lines meeting in a simple elliptic singularity of type $\tilde{E}_7$ and be otherwise nonsingular. Aside from this case, we may assume that Q contains only A-D-E singularities. Then we have:

Proposition 5.11. *Suppose that Q has only A-D-E singularities. Then, up to the notion of equivalence in Definition 5.4, we may assume that B contains precisely two singularities of type $\mathbb{D}_n$, for $n \geq 2$ (here $\mathbb{D}_2$ denotes a pair of pinch points), and is smooth away from the double line.*

Under this assumption, if we denote these two singularities by $\mathbb{D}_m$ and $\mathbb{D}_n$, then:

- *If Q does not contain a line, then $m + n \leq 12$ and all cases occur.*
- *If Q contains a line, then $m = 8$ and $2 \leq n \leq 5$, and all cases occur.*

Proof. By the classification of degenerate cusps above, Q and L may only intersect in pinch points or degenerate cusps of type $\mathbb{D}_n$. As Q and L intersect with multiplicity 4, there are precisely two points of type $\mathbb{D}_n$ on B.

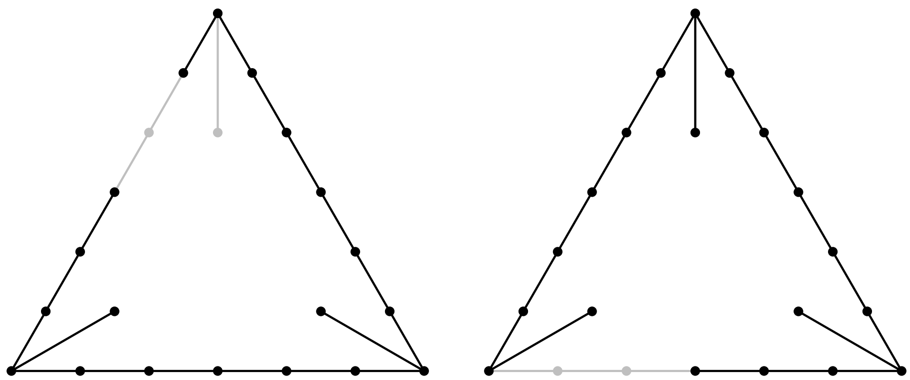
Now, any degenerate cusp of type $\mathbb{D}_n$ arises from either

- a pair of points where Q and L cross transversely (type $\mathbb{D}_2$),
- a point where Q lies tangent to L (type $\mathbb{D}_3$), or
- a point where L crosses a singularity of type A_{n-3} in Q (type $\mathbb{D}_n$ for $n \geq 4$).

Given this, the statement about the singularities arising when Q does not contain a line follows easily from Bruce and Giblin's [BG81, Section 1] enumeration of singular quartics. In particular, we note that if Q contains an A_6 singularity then it must otherwise be smooth, so must intersect L in a $\mathbb{D}_9$ and either a $\mathbb{D}_2$ or $\mathbb{D}_3$; and if Q contains an A_7 singularity it must be a union of two conics that meet only at the singularity, so must intersect L in a $\mathbb{D}_{10}$ and a $\mathbb{D}_2$. Explicit resolutions show that each case forms a distinct equivalence class.

Finally, suppose that Q contains a line. Then Q is the union of a line and a cubic. Resolving, one sees that the only situation not equivalent to one already studied is when the line and cubic meet in a singularity of type A_5 on L, giving rise to a $\mathbb{D}_8$. As the cubic component may be either smooth, nodal or cuspidal, we find that Q and L intersect in a $\mathbb{D}_8$ and a $\mathbb{D}_n$, for $2 \leq n \leq 5$. □

It remains to see how equivalence classes of semistable pairs $(\mathbb{P}^2, B)$ of degree 6 satisfying Assumption 5.2, where $B = 2L + Q$, correspond to hyperbolic subdiagrams of Γ_{21}. Suppose first that Q does not contain a line. Then, by the proposition above, L and Q meet in two singularities, $\mathbb{D}_m$ and $\mathbb{D}_n$, with $m+n \leq 12$. We denote the corresponding subdiagram by $U_{m,n}$; it embeds into Γ_{21} in a way that contains precisely two of the corner vertices, as illustrated in Figure 10a. Here the "tails" on the left and right of the diagram have length m and n vertices respectively (so the



(A) The $U_{4,8}$ subdiagram. (B) The $W_{1,4}$ subdiagram.

FIGURE 10. Subdiagrams of Γ_{21} corresponding to pairs $(\mathbb{P}^2, B)$ where B contains a double line

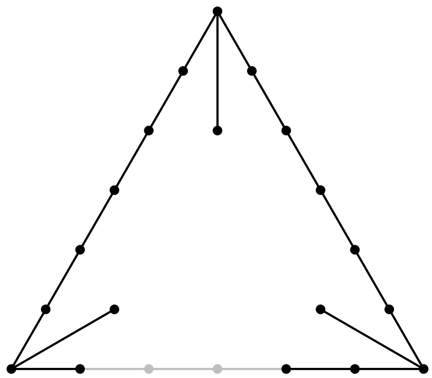
diagram in Figure 10a is $U_{4,8}$). Note that this diagram is parabolic if $(m, n) = (2, 2)$ and hyperbolic otherwise, corresponding to the fact that $(\mathbb{P}^2, B)$ is of Type II if L and Q meet in two singularities of type $\mathbb{D}_2$ and of Type III otherwise.

Next, suppose that Q contains a line but still has at worst A-D-E singularities. Then, by the proposition above, L and Q meet in two singularities, $\mathbb{D}_8$ and $\mathbb{D}_n$, with $n \leq 5$. For consistency with later notation, we denote the corresponding hyperbolic subdiagram by $W_{1,n}$; it embeds into Γ_{21} in a way that contains all three of the corner vertices, as shown in Figure 10b. Here the "tail" on the right of the diagram has a length of n vertices (so the diagram shown in Figure 10b is $W_{1,4}$).

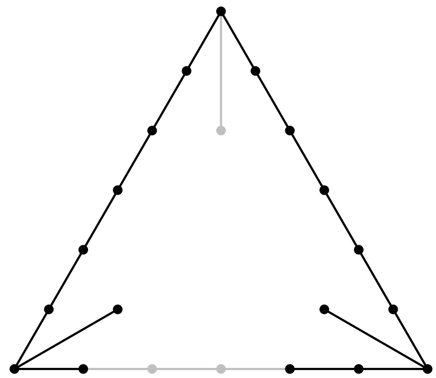
Finally, consider the case where Q contains an $\tilde{E}_7$ singularity. Then Q and L meet in two singularities of type $\mathbb{D}_2$ and $(\mathbb{P}^2, B)$ is of Type II. The corresponding subdiagram is the parabolic subdiagram given by simultaneously embedding a subdiagram of type $T_{4,4}$ and a subdiagram of type $U_{2,2}$ into Γ_{21} without them intersecting.

5.5.2. *Two double lines.* Next suppose that B consists of two double lines, $2L_1$ and $2L_2$, that meet transversely, along with a (possibly reducible) conic C. This conic meets each L_i in a singularity of type $\mathbb{D}_n$. It is easy to see that the two singularities occurring must be either $(\mathbb{D}_2, \mathbb{D}_2)$, $(\mathbb{D}_2, \mathbb{D}_3)$, $(\mathbb{D}_2, \mathbb{D}_4)$, or $(\mathbb{D}_3, \mathbb{D}_3)$, each case forms a single equivalence class, and $(\mathbb{P}^2, B)$ is of Type III in all cases. We denote the hyperbolic subdiagram corresponding to a pair $(\mathbb{D}_m, \mathbb{D}_n)$ by $W_{m,n}$; it embeds into Γ_{21} in a way that contains all three of the corner vertices, as shown in Figure 11a. Here the "tails" on the left and right of the diagram have length m and n vertices respectively (so the diagram pictured in Figure 11a is $W_{2,3}$).

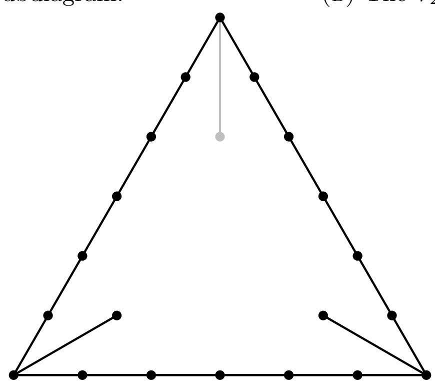
5.5.3. *A double conic.* Next suppose that B consists of a double conic, $2Q$, along with a single conic C. The single conic meets Q in two singularities of type $\mathbb{D}_n$. In a similar way to the previous section, it is easy to see that the two singularities occurring must be either $(\mathbb{D}_2, \mathbb{D}_2)$, $(\mathbb{D}_2, \mathbb{D}_3)$, $(\mathbb{D}_2, \mathbb{D}_4)$, or $(\mathbb{D}_3, \mathbb{D}_3)$, each case forms a single equivalence class, and $(\mathbb{P}^2, B)$ is of Type II if and only if the two singularities are $(\mathbb{D}_2, \mathbb{D}_2)$. We denote the subdiagram corresponding to a pair $(\mathbb{D}_m, \mathbb{D}_n)$ by $V_{m,n}$; it embeds into Γ_{21} in a way that contains precisely two of the corner vertices, as



(A) The $W_{2,3}$ subdiagram.



(B) The $V_{2,3}$ subdiagram.



(C) The Δ_2 subdiagram.

FIGURE 11. Subdiagrams of Γ_{21} corresponding to pairs $(\mathbb{P}^2, B)$ where B contains two double lines, a double conic or a double cubic

shown in Figure 11b. Here the "tails" on the left and right of the diagram have length m and n vertices respectively (so the diagram pictured in Figure 11b is $V_{2,3}$). Note that $V_{2,2}$ is parabolic, whilst the other $V_{p,q}$ are all hyperbolic.

5.5.4. *A double cubic.* Finally, suppose that B is a (possibly singular) double cubic. By the classification of degenerate cusps given in Section 5.5, we see that B must have at worst nodes as singularities. There are thus four possibilities; either B is a smooth double cubic, a double nodal cubic, a double line and double conic, or three double lines, with each forming a distinct equivalence class under Definition 5.4.

The corresponding subdiagrams consist of the entire outer edge of Γ_{21}, along with 0, 1, 2 or 3 of the corner vertices respectively (here the number of corner vertices corresponds to the number of nodes in the double cubic). We denote these subdiagrams by Δ_0, Δ_1, Δ_2 and Δ_3 respectively. Δ_2 is illustrated in Figure 11c. Note that Δ_0 is parabolic and the other Δ_p are hyperbolic.

5.6. **Finishing the proof of Theorem 5.5.** In the previous sections, we have seen that every equivalence class of semistable pairs $(\mathbb{P}^2, B)$ of degree 6 satisfying Assumption 5.2 uniquely determines a subdiagram Γ of Γ_{21}, such that every connected component of Γ is either parabolic or hyperbolic.

Furthermore, this correspondence works both ways: by inspection, every subdiagram of Γ_{21} whose connected components are all parabolic/hyperbolic determines an equivalence class of semistable pairs $(\mathbb{P}^2, B)$ of degree 6 satisfying Assumption 5.2.

In addition it is easy to see that, if a subdiagram has a parabolic connected component, then all connected components are parabolic and the corresponding pair $(\mathbb{P}^2, B)$ is of Type II (in fact, the only disconnected cases are $T_{4,4} \cup U_{2,2}$ and $T_{3,6} \cup T_{3,6}$), and if a subdiagram has a hyperbolic connected component, then that must be the only connected component and the corresponding pair $(\mathbb{P}^2, B)$ is of Type III. Finally, one can easily check that degeneration of pairs $(\mathbb{P}^2, B)$ corresponds to inclusion of the corresponding diagrams. This completes the proof of Theorem 5.5.

6. Family of stable pairs over $\overline{F}_2^{\rm refl}$

Recall from Section 2.2 that the toroidal compactification $\overline{F}_2^{\rm refl}$ is modeled on a single toric variety $U^{\rm refl}$. The corresponding cone $\check\sigma$ in the lattice M_2 of monomials is generated by the 24 vectors $\vec{a}_i, \vec{b}_i, \vec{d}_i' = \frac{1}{2}\vec{d}_i$. In the lattice N_2 of 1-parameter subgroups of the torus (where the fan lives) we have the dual cone with 24 facets and 535 rays.

In this section, we derive sufficient conditions for a family of stable pairs $\big(X, (\frac12 + \epsilon)B\big)$ and their double covers $(S, 2\epsilon D)$ over an open neighborhood $U \ni 0$ of the vertex $0 \in U^{\rm refl}$ to define a map $U \to \overline{F}_2^{\rm refl}$ which is étale over a neighborhood of 0. Such a family would give an extension of the universal family of K3 pairs $(S, 2\epsilon D)$ over the stack $\mathcal{F}_2$ to the toroidal compactification $\overline{\mathcal{F}}_2^{\rm refl}$. We conclude by constructing an explicit candidate family that comes very close to satisfying these conditions.

6.1. **Equations of $U^{\rm refl}$.** Our first task is to give a more explicit description $U^{\rm refl}$ by equations.

Lemma 6.1. *The affine toric variety $U^{\rm refl}$ is the normalization of* Spec A, *where $A = \mathbb{C}[a_i, b_i, d_i']/\sqrt{I}$, and the ideal I generated by the multiplicative versions of Type II relations of Section 2.3, that is:*

$$2\widetilde{E}_8\widetilde{A}_1: \quad a_0^6a_1^4a_2^2b_0^3a_{17}^5a_{16}^4a_{15}^3a_{14}^2a_{13} = a_6^6a_5^4a_4^2b_6^3a_7^5a_8^4a_9^3a_{10}^2a_{11} = b_{12}{d_3'}^2$$
$$\widetilde{D}_{10}\widetilde{E}_7: \quad a_{17}b_0(a_0\cdots a_6)^2b_6a_7 = a_{12}^4b_{12}^2a_{11}^3a_{13}^3a_{10}^2a_{14}^2a_9a_{15}$$
$$\widetilde{D}_{16}\widetilde{A}_1: \quad a_1b_0(a_0\cdots a_6)^2b_6a_5 = a_3{d_3'}^2,$$

etc., by S_3-symmetry.

Proof. This is simply the definition from toric geometry: $U^{\rm refl}$ is the Spec of a semigroup algebra over $\mathbb{C}$; the semigroup is generated by the 24 vectors in the lattice of monomials, and the relations follow from the relations between the vectors. We take the radical of $\sqrt{I}$ and then normalize because we did not check if the the ideal I is radical, and if one has to saturate the ideal of relations. ∎

Lemma 6.2. *Alternatively, U^{refl} is the normalization of* $\operatorname{Spec} A'$*, where A' is a subring of $\mathbb{C}[t_i, s_j]$, where $0 \le i < 18$, $j = 0, 6, 12$ generated by*

$$a_i = \frac{t_{i-1}t_{i+1}}{t_i^2} \ (i \ne 0,6,12), \quad a_i = \frac{t_{i-1}t_{i+1}s_i}{t_i^2} \ (i = 0,6,12),$$
$$b_i = \frac{t_i}{s_i^2} \ (i = 0,6,12), \qquad d'_i = t_i s_{i+9} \ (i = 3,9,15),$$

where the indices i are considered modulo 18.

Proof. This is simply reformulation of the above Lemma using the coordinates $\vec{s}_i, \vec{t}_i$ for the cone $\check{\sigma}$, as in (2.6). Since A' is a subring of a polynomial ring, A' is an integral domain. ☐

6.2. **Monodromy in the toric families.** We are going to write equations for our family over U^{refl} in terms of the coordinates t_i, s_j. Recall from Section 4.2 that we have 4=1+3 toric 16-dimensional subvarieties $U_{\mathcal{T}_0} \subset U^{\mathrm{refl}}$ and for each of them a family over an open subset $U_{\mathcal{T}_0} \supset U \ni 0$ of the origin. Two of these toric subvarieties are

(1) For the 6-6-6 triangle of Figure 3b. It is defined by setting $b_0 = b_6 = b_{12} = 1$, in other words $t_i = s_i^2$ for $i = 0, 6, 12$.
(2) For the 3-3-12 triangle of Figure 4. It is defined by setting $b_0 = b_6 = d'_3 = 1$, in other words $t_i = s_i^2$ for $i = 0, 6$ and $t_3 s_{12} = 1$.

The last two toric subvarieties are obtained by applying the S_3-action to (2).

We denote the corresponding sublattices of N_2 by $N^0 = \langle \vec{b}_0, \vec{b}_6, \vec{b}_{12}\rangle^\perp$, $N^1 = \langle \vec{b}_0, \vec{b}_6, \vec{d}'_3\rangle^\perp$, $N^2 = \langle \vec{b}_6, \vec{b}_{12}, \vec{d}'_9\rangle^\perp$, $N^3 = \langle \vec{b}_{12}, \vec{b}_0, \vec{d}'_{15}\rangle^\perp$.

Consider a one-parameter subfamily $\mathbb{A}^1 \to U_{\mathcal{T}_0}$ for one of the toric families. This defines a 1-parameter degeneration of polarized K3 surfaces $\big(S, \mathcal{O}(D)\big)$. Associated with it, there is a monodromy vector $n \in N_2$ defined up to $O(N_2)$. We recall the following fundamental result of Friedman and Scattone [FS86]:

Theorem 6.3 (Friedman-Scattone)**.** *Let $\mathcal{S}^{\mathrm{Kul}} \to \mathbb{A}^1$ be any Kulikov model of $\mathcal{S} \to \mathbb{A}^1$, that is: $\mathcal{S}$ is smooth, the central fiber is a reduced divisor with normal crossings, and $\omega_{\mathcal{S}/\mathbb{A}^1}$ is trivial. Then the number of triple points in the central fiber $\mathcal{S}_0^{\mathrm{Kul}}$ equals $n^2 \in 2\mathbb{Z}$.*

This theorem allows to compute the induced monodromy map $N^i \to N_2$ for each of the toric varieties, by computing the number of triple points for each 1-parameter subfamily.

Lemma 6.4. *Consider a one-parameter toric degeneration of a pair $(\mathbb{P}^2, B_6)$ defined by a subdivision of the 6-6-6 triangle given by a choice of heights $T_0, \dots, T_{17}$, T_c as in Section 4.3. Then the number of triple points in a Kulikov model of the corresponding one-parameter degeneration of K3 surfaces of degree 2 equals F_3^{toric}, where*

$$\frac{1}{4}F_3^{\mathrm{toric}} = \sum_{i=0}^{17}\left(-2T_i^2 + 2T_iT_{i+1}\right) + \frac{1}{2}\left(T_0^2 + T_6^2 + T_{12}^2\right) + T_c\left(\frac{3}{2}T_c - T_0 - T_6 - T_{12}\right)$$

Proof. Let $M = \mathbb{Z}^2$ be the lattice of monomials of the toric variety $\mathbb{P}^2$, and let P in $M \oplus \mathbb{Z} = \mathbb{Z}^3$ be the polyhedron corresponding to the one-parameter degeneration X. As we explained in Section 4.3, P is the convex hull of the sets $v_i + \mathbb{R}_{\ge 0}(0,0,1) \in \mathbb{Z}^3$, where $v_i = (\vec{u}_i, T_i)$, and $\vec{u}_i$ were defined in Theorem 2.6, $i \in \{0, \dots, 17, c\}$.

Let P^+ be a polyhedron obtained by cutting P by a horizontal plane near the vertex $(\vec{u}_c, T_c)$. It corresponds to a blowup X^+ of X. The central fiber X_0^+ is given by the surface of the lower convex hull of P^+. We assume that a divisible enough ramified base change $s = t^d$ was made on the base S of $X \to S$, so that X_0^+ is reduced.

At every interior vertex q of P^+ (i.e. lying on the cutting plane), exactly 3 polygons meet, and the corresponding dual cone in the lattice $N \oplus Z$ is simplicial and is generated by the 3 vectors

$$(v_{i-1} - v_c) \times (v_i - v_c),\ (v_i - v_c) \times (v_{i+1} - v_c),\ (0,0,1).$$

(Here $u \times v$ stands for the cross product of two vectors in $\mathbb{R}^3$). Thus, X^+ has a cyclic toric singularity at the corresponding point. A resolution of X^+ replaces this singularity by $\det_i$ normal crossing triple points, where $\det_i$ is the determinant of the 3×3 matrix given by the above three vectors. Explicitly,

$$\begin{aligned}\det_0 &= -3T_0^2 + 2T_0T_{17} + 2T_0T_1 + T_c(2T_0 - 2T_1 - 2T_{17} + T_c)\\ \det_1 &= -4T_1^2 + 2T_1T_0 + 2T_1T_2 + T_c(4T_1 - 2T_0 - 2T_2),\end{aligned}$$

etc., by symmetry. The number of triple points F_3^{toric} is the sum of these numbers, multiplied by 2 because of the double cover from the family of K3s to the family of $\mathbb{P}^2$'s. This gives the formula. □

Remark 6.5. Note that $\sum_{i=0}^{17}(-2T_i^2 + 2T_iT_{i+1}) = -\sum_{i=0}^{17}(T_i - T_{i+1})^2$.

Remark 6.6. This computation is similar to a computation made by Adrian Brunyate [Bru15] in the case of elliptic K3 surfaces.

One the other hand, every vector n in the lattice $L_2 = E_8^2 \oplus U$ can be written as

$$n = \sum_{i=0}^{17} T_i\vec{a}_i + (S_0\vec{b}_0 + S_6\vec{b}_6 + S_{12}\vec{b}_{12}).$$

because by (2.5) the vectors $\vec{a}_0, \dots, \vec{a}_{17}, \vec{b}_0, \vec{b}_6, \vec{b}_{12}$ generate N_2.

Lemma 6.7. *One has*

$$n^2 = \sum_{i=0}^{17}\left(-2T_i^2 + 2T_iT_{i+1}\right) - 2\left(S_0^2 + S_6^2 + S_{12}^2\right) + 2\left(T_0S_0 + T_6S_6 + T_{12}S_{12}\right)$$

and

$$4n^2 = F_3^{\text{toric}} - 2\sum_{i=0,6,12}(T_i - T_c - 2S_i)^2 - 8T_c\sum_{i=0,6,12} S_i.$$

Proof. Immediate calculation. □

Corollary 6.8. *When $T_c = 0$ and $S_i = \frac{1}{2}T_i$ for $i = 0, 6, 12$, one has $(2n)^2 = F_3^{\text{toric}}$.*

Remark 6.9. Recall that

$$\begin{aligned} n \cdot \vec{a}_i &= T_{i-1} - 2T_i + T_{i+1} + S_i \quad (S_i \text{ appear only for } i = 0,6,12),\\ n \cdot \vec{b}_i &= T_i - 2S_i \quad (i = 0,6,12),\\ n \cdot \vec{d}_i &= T_i + S_{i+9} \quad (i = 3,9,15).\end{aligned}$$

Thus, after setting $T_c = 0$, the condition $S_i = \frac{1}{2}T_i$ is equivalent to the condition $n \perp \vec{b}_i$, and the formula above becomes

$$(2n)^2 = F_3^{\text{toric}} - 2 \sum_{i=0,6,12} (n \cdot \vec{b}_i)^2.$$

We may perform similar computations for the $\mathbb{F}_4^0$ toric degeneration corresponding to the 3-3-12 triangle and the monodromy vector

$$n = \sum_{i=0}^{17} T_i \vec{a}_i + S_0 \vec{b}_0 + S_6 \vec{b}_6 + R_3 \vec{d}_3.$$

In that case, we find:

Lemma 6.10. *One has*

$$\begin{aligned} n \cdot \vec{a}_i &= T_{i-1} - 2T_i + T_{i+1} + S_i \quad (S_i \textit{ appear only for } i = 0, 6), \\ n \cdot \vec{b}_i &= T_i - 2S_i \quad (i = 0, 6), \\ n \cdot \vec{d}_3 &= 2T_i - 2R_3 \end{aligned}$$

and

$$\begin{aligned} 4n^2 = G_3^{\text{toric}} \quad &- \quad 2(T_0 - T_c - 2S_0)^2 - 2(T_6 - T_c - 2S_6)^2 - 2(T_3 - T_c - R_3)^2 \\ &- \quad 8T_c\,(S_0 + S_6 + 2R_3) \end{aligned}$$

Corollary 6.11. *If $T_c = 0$ and $n \perp \langle \vec{b}_0, \vec{b}_6, \vec{d}_3 \rangle$ then one has $(2n)^2 = G_3^{\text{toric}}$.*

Corollaries 6.8 and 6.11 together give the following theorem.

Theorem 6.12. *For each of the 1+3 toric families, the monodromy of the standard toric family is consistent with the induced monodromy map $N^i \to N_2$ being the standard inclusion $N^i \to N_2$.*

Proof. We only need to explain the appearance of the extra coefficient 4 in the formulas $(2n)^2 = F_3^{\text{toric}}$, $(2n)^2 = G_3^{\text{toric}}$. The families above are families of the surfaces $X = \mathbb{P}^2$ or $\mathbb{F}_4^0$ which appear as bases of the double covers $S \to X$. The families of the K3 surfaces X are constructed on the square roots of these families. This gives an extra coefficient 4. □

6.3. **Sufficient conditions for extending the universal family of K3s.** We are now prepared to write an explicit family of stable pairs $\big(X, (\frac{1}{2} + \epsilon)B\big)$ over an open subset $U \ni 0$ of the affine toric variety U^{refl} which defines an extension of the family of degree 2 K3 surfaces over the moduli stack $\overline{\mathcal{F}}_2$.

Theorem 6.13. *Suppose that we are given a family of pairs $\big(\mathcal{X}, (\frac{1}{2} + \epsilon)\mathcal{B}\big) \to U^{\text{refl}}$ that satisfies the following conditions:*

1. *(1) Over some open subset U of the origin $0 \in U^{\text{refl}}$, it is a family of stable pairs of the types described in Theorem 4.1.*
2. *(2) Over each of the 1+3 toric families it coincides with the standard toric families.*

Then this family defines a morphism $U' \to \overline{\mathcal{F}}_2^{\text{refl}}$ from a square root of U to the moduli stack of polarized degree 2 K3 surfaces, and this morphism is étale in a neighborhood of the origin $0 \in U'$, resp. the maximal degeneration point in $\overline{\mathcal{F}}_2^{\text{refl}}$.

Moreover, this family gives an extension of the universal family of K3 surfaces over $\mathcal{F}_2$ to a family of stable pairs $\big(X, (\frac{1}{2} + \epsilon)B\big)$ over $\overline{\mathcal{F}}_2^{\text{refl}}$.

Proof. The family defines a *rational* map $U' \to \overline{\mathcal{F}}_2^{\mathrm{refl}}$. It also induces a monodromy homomorphism $\Phi\colon N_2 \to N_2$. We first claim that Φ is an isomorphism. This follows from the fact that the pullback of the intersection form under Φ to each of the sublattices $N^i \subset N_2$ for the 1+3 toric families coincides with the intersection form on N^i by Theorem 6.12, and that these 4 families span the lattice N_2 and have sufficiently large intersections.

The rational map $U' \to \overline{\mathcal{F}}_2^{\mathrm{refl}}$ above is regular if the cone $\check{\sigma}$ maps into a cone in the fan τ^{refl}, and it is étale if it maps isomorphically to a cone in τ^{refl}. But this is true by construction: under an isometry Φ, the cone $\check{\sigma}$ simply maps to a copy of $\check{\sigma}$ in τ^{refl}, where it is a unique up to an $O(N_2)$ maximal cone. So the induced map is étale.

This gives an extension of the universal family of K3 surfaces to an open neighborhood of the 0-cusp. To have an extension to the whole $\overline{F}_2^{\mathrm{refl}}$, we also have to construct extensions to open neighborhoods of the 1-cusps for the Type II degenerations. But with our descriptions of the corresponding stable surfaces, cf. Example 4.7, this part is straightforward. □

6.4. **Explicit family.** We start with the system of equations considered in Section 5.1.1. Treating these as parametric equations, as in Section 4.3, we may define a family as $\operatorname{Proj} R \to \operatorname{Spec} A$, where $A \subset \mathbb{C}[t_i^{\pm}, s_j^{\pm}]$ is the normalization of the subalgebra generated by the variables a_i, b_i, d_i', i.e. $\operatorname{Spec} A = U^{\mathrm{refl}}$, and R is the graded subalgebra of $A[x,y,z]$ generated by the 19 polynomials u_i given in in Section 5.1.1.

To simplify the equations defining this system, we recast it in a weighted projective space, where the variables have the following degrees:

degree	variables
1	$x, y, z,\ u_0, u_6, u_{12}$
2	u_3, u_9, u_{15}
3	$u_2, u_4, u_8, u_{10}, u_{14}, u_{16}$
6	$u_1, u_5, u_7, u_{11}, u_{13}, u_{17}$

Note that this gives rise to fractional powers of the t_i's, such as $t_0^{1/6}x$, but in the variables of degree 6, such as t_0x^6, these roots disappear. The resulting family is given by:

$$
\begin{aligned}
&u_c = -xyz + s_0x^3 + s_6y^3 + s_{12}z^3 \\
&u_0 = t_0^{\frac16}x && u_1 = t_1x^4(-xy+s_{12}z^2) && u_2 = t_2^{\frac12}x(-xy+s_{12}z^2)\\
&u_3 = t_3^{\frac13}(-xy+s_{12}z^2) && u_4 = t_4^{\frac12}y(-xy+s_{12}z^2) && u_5 = t_5y^4(-xy+s_{12}z^2)\\
&u_6 = t_6^{\frac16}y && u_7 = t_7y^4(-yz+s_0x^2) && u_8 = t_8^{\frac12}y(-yz+s_0x^2)\\
&u_9 = t_9^{\frac13}(-yz+s_0x^2) && u_{10} = t_{10}^{\frac12}z(-yz+s_0x^2) && u_{11} = t_{11}z^4(-yz+s_0x^2)\\
&u_{12} = t_{12}^{\frac16}z && u_{13} = t_{13}z^4(-xz+s_6y^2) && u_{14} = t_{14}^{\frac12}z(-xz+s_6y^2)\\
&u_{15} = t_{15}^{\frac13}(-xz+s_6y^2) && u_{16} = t_{16}^{\frac12}x(-xz+s_6y^2) && u_{17} = t_{17}x^4(-xz+s_6y^2)
\end{aligned}
$$

We can further simplify this family by introducing three new variables u, v, w of degree two, as follows:

$$\begin{array}{lll}
u = -yz + s_0x^2 & & \\
v = -xz + s_6y^2 & & \\
w = -xy + s_{12}z^2 & & \\
u_c = 2xyz + xu + yv + zw & & \\
u_0 = t_0^{\frac{1}{6}}x & u_1 = t_1x^4w & u_2 = t_2^{\frac{1}{2}}xw \\
u_3 = t_3^{\frac{1}{3}}w & u_4 = t_4^{\frac{1}{2}}w & u_5 = t_5y^4w \\
u_6 = t_6^{\frac{1}{6}}y & u_7 = t_7y^4u & u_8 = t_8^{\frac{1}{2}}yu \\
u_9 = t_9^{\frac{1}{3}}u & u_{10} = t_{10}^{\frac{1}{2}}zu & u_{11} = t_{11}z^4u \\
u_{12} = t_{12}^{\frac{1}{6}}z & u_{13} = t_{13}z^4v & u_{14} = t_{14}^{\frac{1}{2}}zv \\
u_{15} = t_{15}^{\frac{1}{3}}v & u_{16} = t_{16}^{\frac{1}{2}}xv & u_{17} = t_{17}x^4v
\end{array}$$

We would like to apply Theorem 6.13 to this family, but condition (2) of that theorem is not satisfied. To rectify this, we make two further changes. Firstly, we bring this family into agreement with the toric family corresponding to the 6-6-6 triangle by making the substitution $s_j \mapsto s_j - t_j^{\frac{1}{2}}$ for each $j \in \{0, 6, 12\}$. Then, to achieve agreement with the three toric families corresponding to the 3-3-12 triangle, we introduce coefficients to u, v, w which vanish when the corresponding $d_i = 0$, as follows:

$$\begin{array}{lll}
(1 - s_0^2t_9^2)u = -yz + (s_0 - t_0^{\frac{1}{2}})x^2 & & \\
(1 - s_6^2t_{15}^2)v = -xz + (s_6 - t_6^{\frac{1}{2}})y^2 & & \\
(1 - s_{12}^2t_3^2)w = -xy + (s_{12} - t_{12}^{\frac{1}{2}})z^2 & & \\
u_c = 2xyz + xu + yv + zw & & \\
u_0 = t_0^{\frac{1}{6}}x & u_1 = t_1x^4w & u_2 = t_2^{\frac{1}{2}}xw \\
u_3 = t_3^{\frac{1}{3}}w & u_4 = t_4^{\frac{1}{2}}yw & u_5 = t_5y^4w \\
u_6 = t_6^{\frac{1}{6}}y & u_7 = t_7y^4u & u_8 = t_8^{\frac{1}{2}}yu \\
u_9 = t_9^{\frac{1}{3}}u & u_{10} = t_{10}^{\frac{1}{2}}zu & u_{11} = t_{11}z^4u \\
u_{12} = t_{12}^{\frac{1}{6}}z & u_{13} = t_{13}z^4v & u_{14} = t_{14}^{\frac{1}{2}}zv \\
u_{15} = t_{15}^{\frac{1}{3}}v & u_{16} = t_{16}^{\frac{1}{2}}xv & u_{17} = t_{17}x^4v
\end{array}$$

Then we have:

Lemma 6.14. *When restricted to the subfamilies*

(1) $t_0 = s_0^2$, $t_6 = s_6^2$, $t_{12} = s_{12}^2$,
(2) $t_0 = s_0^2$, $t_6 = s_6^2$, $t_3s_{12} = 1$, *(and its images under S_3-symmetry),*

over an open subset U of the origin $0 \in U^{\mathrm{refl}} \supset U$, this family gives the standard toric families, as in Section 4.3.

Proof. In case (1), near the origin the variables u, v, w are simply nonzero multiples of yz, xz and xy respectively. So the (powers) of the variables $u_0, \dots, u_{17}, u_c$ are just the monomials from the toric family of Section 4.3.

In case (2), the third parametric equation makes the surface into a cone $\mathbb{F}_4^0$ (considering x, y, z, w as coordinates on $\mathbb{P}(1,1,1,2)$, this is just the image of a Veronese embedding $\mathbb{P}(1,1,4) \hookrightarrow \mathbb{P}(1,1,1,2)$). Then, by observation, the other monomials become nonzero multiples of the corresponding boundary and center monomials in the 3-3-12 triangle. □

To apply Theorem 6.13, we need to check whether the fibers of this family are stable pairs $\big(X, (\frac{1}{2}+\epsilon)B\big)$. It suffices to check this on each of the 103 types of divisors on $U^{\rm refl}$, corresponding to the 103 types (modulo S_3) of rays of $\check\sigma^{\rm refl}$. Each ray gives a system of heights, defined uniquely up to a multiple: $t_i = t^{T_i}$, $s_j = t^{S_j}$. A direct computation (which in the easy cases was done by hand and in more complicated cases was performed by *Singular*) then shows that in 99 out of 103 cases the answer is consistent with the stable pairs described in Theorem 4.1, but in the remaining 4 cases (corresponding to very long A'_{18}, D_{18}, $A'_{16}A_1^c$ and $D_{17}A_0$ subdiagrams of $\Gamma_{\rm Vin}$) the limits of the one-parameter degenerations are unstable.

In light of this, it seems reasonable to expect that a family of stable pairs satisfying Theorem 6.13 does exist, perhaps given by further deforming the families constructed here by the addition of terms of higher order in the t_i, s_j. However, computational limits have prevented us from searching for such a family. We therefore make the following conjecture:

Conjecture 6.15. *There exists a family of pairs* $\big(\mathcal{X}, (\frac{1}{2}+\epsilon)\mathcal{B}\big) \to U^{\rm refl}$ *satisfying the conditions of Theorem 6.13. This gives rise to an extension of the universal family of K3 surfaces over* $\mathcal{F}_2$ *to a family of stable pairs* $\big(X, (\frac{1}{2}+\epsilon)B\big)$ *over* $\overline{\mathcal{F}}_2^{\rm refl}$.

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Appendix

In this appendix we give a list of all 103 maximal (i.e. rank 18) subdiagrams of Γ_{Vin}, up to equivalence 4.6. The subdiagrams are listed in the second column of the table; here we use the subdiagram naming convention from Table 3, omitting any subdiagrams supported on internal vertices, but including A_0 subdiagrams for $\mathbb{F}_2^0$ components. There is thus a bijection between the components of subdiagrams listed in the second column and the components in the corresponding stable limit. The third column gives the number of isolated internal vertices (b_i, d_i) in the subdiagram; by equivalence 4.6 these may be ignored in our theory, but are needed to uniquely reconstruct the subdiagrams. Cases 1-99 are elliptic subdiagrams corresponding to Type III degenerations, and cases 100-103 are parabolic subdiagrams corresponding to Type II degenerations.

The final column gives a construction for the stable limits corresponding to these subdiagrams. If this column contains the word "Toric" (resp. "Unigonal"), then the degeneration may be constructed using the method of Section 4.2 applied to a subdivision of the 6-6-6 triangle (resp. 3-3-12 triangle). Otherwise, this column describes how to construct the degeneration; this consists of either a description of a series of elementary modifications that may be performed on a Type II degeneration to obtain the required degeneration, or a reference to Table 5 or Section 4.4.4, where more difficult cases are considered separately.

No. #	Subdiagram	Int. Vert.	Construction
1	A'_{18}	0	4 type 2 mod. on A''_{15} in $A''_{15}A^c_3$ (#8)
2	D_{18}	0	2 type 2 mod. on $\tilde{D}_{16}$ in $\tilde{D}_{16}\tilde{A}_1$
3	A^c_{17}	1	Type 1 mod. on $\tilde{D}_{16}$ in $\tilde{D}_{16}\tilde{A}_1$
4	$A'_{16}A^c_1$	1	2 type 2 mod. on A''_{15} in $A''_{15}A^c_3$ (#8)
5	$D_{17}A_0$	1	Type 2 mod. on $\tilde{D}_{16}$ in $\tilde{D}_{16}\tilde{A}_1$
6	$D'_{16}A_2$	0	See Section 4.4.4
7	$D'_{16}A^c_1A_0$	1	Unigonal
8	$A''_{15}A^c_3$	0	Type 1 mod. on $\tilde{A}_1$ in $\tilde{D}_{16}\tilde{A}_1$
9	$A''_{15}A^c_1A^c_1$	1	Unigonal
10	A^e_{17}	1	4 type 1 mod. on $\tilde{D}_{10}$ in $\tilde{D}_{10}\tilde{E}_7$
11	$A'_{13}A_4$	1	See Table 5
12	$D_{14}A^e_3$	1	4 type 2 mod. on $\tilde{D}_{10}$ in $\tilde{D}_{10}\tilde{E}_7$
13	$A_{14}A_2$	2	See Table 5
14	$A'_{13}A_2A^c_1$	2	Unigonal
15	$D_{14}A_2A_0$	2	Unigonal
16	$A'_{12}A'_6$	0	See Table 5
17	$D_{13}A'_5$	0	3 type 2 mods. on $\tilde{D}_{10}$ in $\tilde{D}_{10}\tilde{E}_7$
18	$A^c_{13}A'_4$	1	See Table 5
19	$A'_{12}A'_4A^c_1$	1	Unigonal
20	$D_{13}A'_4A_0$	1	Unigonal
21	$A'_{11}D_7$	0	See Table 5
22	$D_{12}D_6$	0	2 type 2 mod. on $\tilde{D}_{10}$ in $\tilde{D}_{10}\tilde{E}_7$
23	$A_{12}D_5$	1	See Table 5
24	$A'_{11}D_5A^c_1$	1	Unigonal

25	$D_{12}D_5A_0$	1	Unigonal
26	$A'_{10}E_8$	0	See Table 5
27	$A^c_9E_8$	1	See Table 5
28	$D_{11}E_7$	0	Type 2 mod. on $\tilde{D}_{10}$ in $\tilde{D}_{10}\tilde{E}_7$
29	$A_{10}E_7$	1	See Table 5
30	$A^c_{11}E_6$	1	Type 1 mod. on $\tilde{D}_{10}$ in $\tilde{D}_{10}\tilde{E}_7$
31	$A'_{10}E_6A^c_1$	1	Unigonal
32	$D_{11}E_6A_0$	1	Unigonal
33	$D'_{10}A_8$	0	See Table 5
34	$D'_{10}E_7A^c_1$	1	Toric
35	$A'_9E_7A^c_1$	1	Unigonal
36	$D_{10}E_7A_0$	1	Unigonal
37	$D'_{10}D_6A_2$	0	Toric
38	$D'_{10}A'_5A^c_3$	0	Toric
39	$D'_{10}A_4A^e_3$	1	Toric
40	$D'_{10}A'_6A_2$	0	Toric
41	$D'_{10}D_7A^e_1$	0	Toric
42	$D'_{10}E_8A_0$	0	Toric
43	$D_9E_8A_0$	1	Unigonal
44	$A^c_9A''_9$	0	Type 1 mod. on $\tilde{E}_7$ in $\tilde{D}_{10}\tilde{E}_7$
45	$A''_9E_8A^c_1$	0	Toric
46	$A'_8E_8A^c_1$	1	Unigonal
47	$A''_9D_7A_2$	0	Toric
48	$A''_9A'_6A^c_3$	0	Toric
49	$A''_9A_4A_4$	1	Toric
50	$A_{10}A'_7$	1	See Table 5
51	$A^e_9D_8$	1	See Table 5
52	$D_8A'_7A_2$	1	Toric
53	$D_8D_8A^e_1$	1	Toric
54	$A'_7A'_7A^c_3$	1	Toric
55	$A^e_{11}A^e_5$	2	2 type 1 mod. on $\tilde{E}_7$ in $\tilde{D}_{10}\tilde{E}_7$
56	$A'_7A^e_5A_4$	2	Toric
57	$D_8A^e_5A^e_3$	2	Toric
58	$A'_7A'_6A_4$	1	Toric
59	$D_8A'_5A_4$	1	Toric
60	$D_7A'_7A^e_3$	1	Toric
61	$D_8D_6A^e_3$	1	Toric
62	$E_8A'_7A_2$	1	Toric
63	$E_8A_6A_2$	2	Unigonal
64	$D_8E_7A_2$	1	Toric
65	$A^e_7E_7A_2$	2	Unigonal
66	$A_8E_6A_2$	2	Unigonal
67	$A^e_9A_2D_5$	2	Unigonal
68	$A_{10}A'_4A_2$	2	Unigonal
69	$A^e_{11}A_2A_2$	3	Unigonal
70	$A'_6A'_6A'_6$	0	Toric
71	$D_7A'_6A'_5$	0	Toric
72	$E_8A'_6A'_4$	0	Toric

73	$E_8 A_5^c A_4'$	1	Unigonal
74	$D_7 D_6 A_5'$	0	Toric
75	$E_8 D_5 A_5'$	0	Toric
76	$E_8 D_5 A_4$	1	Unigonal
77	$D_7 E_7 A_4'$	0	Toric
78	$E_7 A_6 A_4'$	1	Unigonal
79	$E_8 E_6 A_4'$	0	Toric
80	$E_8 E_6 A_3^c$	1	Unigonal
81	$A_7^c E_6 A_4'$	1	Unigonal
82	$A_8 D_5 A_4'$	1	Unigonal
83	$A_9^c A_4' A_4'$	1	Unigonal
84	$E_8 E_7 A_3'$	0	Toric
85	$E_8 E_7 A_2$	1	Unigonal
86	$E_8 E_8 A_2'$	0	Toric
87	$E_8 E_8 A_1^c$	1	Unigonal
88	$D_6 D_6 D_6$	0	Toric
89	$E_7 D_6 D_5$	0	Toric
90	$E_7 A_5^e D_5$	1	Unigonal
91	$E_7 E_6 D_5$	0	Toric
92	$E_7 E_6 A_4$	1	Unigonal
93	$A_6 E_6 D_5$	1	Unigonal
94	$A_7^e D_5 D_5$	1	Unigonal
95	$E_7 E_7 D_4$	0	Toric
96	$E_7 E_7 A_3^e$	1	Unigonal
97	$E_6 E_6 E_6$	0	Toric
98	$E_6 E_6 A_5^c$	1	Unigonal
99	$A_5^e A_5^e A_5^e$	3	Toric
100	$\tilde{D}_{16} \tilde{A}_1$	0	Unigonal
101	$\tilde{E}_8 \tilde{E}_8$	2 ($\tilde{A}_1$)	Toric
102	$\tilde{D}_{10} \tilde{E}_7$	0	Toric
103	$\tilde{A}_{17}$	0	Cubic cone

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