The 14th Case VHS And K3 Fibrations III: Elliptic Surfaces

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This talk is the third in a series of three given by Charles Doran, Andrey Novoseltsev and myself at the CMS Winter Meeting 2012. It is based upon the material in Section 7 of the preprint [CDL$^+$].

In the previous talks, we saw an explicit construction of a family of threefolds that realises a variation of Hodge structure that was conjectured to exist in [DM06]. In this talk we will see a second method for constructing this family, using elliptic surfaces.

We begin by recalling the basic setup. W is a Calabi-Yau threefold with a single node that admits a fibration $W \to \mathbb{P}^1$ by Kummer surfaces $\mathrm{Kum}(E_1 \times E_2)$. This W is a general member of the family of Calabi-Yau threefolds that realise the fourteenth case variation of Hodge structure from [DM06] mentioned above. Our aim is to find a direct method by which W can be constructed.

The proof of [CD11, Theorem 3.13] shows that there is a canonical choice of sixteen (-2)-curves in a general fibre of $W \to \mathbb{P}^1$, which must therefore sweep out a number of divisors on W. Take a double cover of W ramified over these divisors and contract the (-1)-curves that result. This undoes the Kummer construction, giving a new threefold $\mathcal{A}$ that is fibred over $\mathbb{P}^1$ by products of elliptic curves $E_1 \times E_2$. Furthermore, we have an expression for the j-invariants of E_1 and E_2, they are given by the roots of the quadratic equation

$$j^2 - j + \frac{uv}{(u+v)^2 12^6 \xi_0} = 0,$$

where (u, v) are coordinates on the base $\mathbb{P}^1_{u,v}$ and ξ_0 is a modular parameter. For now, we assume that W is generic so that $\xi_0 \neq 0, \frac{1}{12^6}, \infty$, note that degenerate behaviour occurs at those values of ξ_0.

We would like to construct a birational model for $\mathcal{A}$. Ideally, as the general fibre of $\mathcal{A} \to \mathbb{P}^1_{u,v}$ is a product of elliptic curves, we would like to construct such a model as a product of elliptic surfaces. Unfortunately, however, the equation for the j-invariant above shows that monodromy around one of the two points in $\mathbb{P}^1_{u,v}$

$$(u,v) = \left(2 - 12^6\xi_0 \pm 2\sqrt{1 - 12^6\xi_0}, 12^6\xi_0\right)$$

switches E_1 and E_2, so such a splitting is not possible.

To solve this, let $\bar{f}\colon \mathbb{P}^1_r \to \mathbb{P}^1_{u,v}$ be the double cover of $\mathbb{P}^1_{u,v}$ ramified over the two points above (here $\mathbb{P}^1_r$ is equipped with affine coordinate r). Then $\mathbb{P}^1_r$ also admits a double cover $\bar{g}\colon \mathbb{P}^1_r \to \mathbb{P}^1_j$ of the j-line $\mathbb{P}^1_j$, ramified over the two points

$$j = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{1}{12^6\xi_0}}.$$

We have a diagram

$$\begin{array}{ccc} & \mathbb{P}^1_r & \\ {}^{\bar{g}}\swarrow & & \searrow^{\bar{f}} \\ \mathbb{P}^1_j & & \mathbb{P}^1_{u,v} \end{array}$$

Let $\mathcal{A}'$ denote the threefold obtained by pulling back $\mathcal{A} \to \mathbb{P}^1_{u,v}$ by the morphism f'. Then we have:

Proposition 1. *$\mathcal{A}' \to \mathbb{P}^1_r$ is birational over $\mathbb{P}^1_r$ to a fibre product $\mathcal{E}_1 \times_{\mathbb{P}^1_r} \mathcal{E}_2$ of elliptic surfaces $\mathcal{E}_{1,2} \to \mathbb{P}^1_r$ with section. Furthermore, the j-invariants of the elliptic curves E_1 and E_2 forming the fibres of $\mathcal{E}_1$ and $\mathcal{E}_2$ over a point $r \in \mathbb{P}^1_r$ are given by*

$$j_1 = \bar{g}(r),$$
$$j_2 = 1 - j_1.$$

Thus, in order to construct a birational model for $\mathcal{A}'$, and hence W, it is enough to construct the elliptic surfaces $\mathcal{E}_1$ and $\mathcal{E}_2$. Our job is made even easier by the following lemma:

Lemma 2. *Let $i\colon \mathbb{P}^1_r \to \mathbb{P}^1_r$ be the involution that switches the preimages of a point $(u,v) \in \mathbb{P}^1_{u,v}$ under $\bar{f}$. Then i induces an isomorphism $\mathcal{E}_1 \to \mathcal{E}_2$.*

From this, we see that it is sufficient to construct just the elliptic surface $\mathcal{E}_1$. Furthermore, as we know the j-invariant map for $\mathcal{E}_1$, we already have $\mathcal{E}_1$ up to quadratic twists. To completely determine $\mathcal{E}_1$ we just need to determine the type and location of its singular fibres.

At this point, a canonical bundle calculation shows that $\omega_{\mathcal{E}_1} \cong \mathcal{O}_{\mathcal{E}_1}(-E)$, where E denotes the class of a fibre in $\mathcal{E}_1$. This gives us that the Euler number of $\mathcal{E}_1$ is 12. As $\mathcal{E}_1$ has six singular fibres, the only possible combination of singular fibres on $\mathcal{E}_1$ that can lead to this Euler number is two each of types I_1, II and III occurring at the points over $j = \infty$, 0 and 1 respectively.

This completes the construction of $\mathcal{E}_1$. We can obtain a birational model for W by first constructing $\mathcal{E}_1 \times_{\mathbb{P}^1_r} \mathcal{E}_2$, then quotienting by the involution induced by $i \colon \mathbb{P}^1_r \to \mathbb{P}^1_r$ to obtain a birational model for $\mathcal{A}$, then performing the fibrewise Kummer construction.

We conclude with a brief discussion of the degenerate case $\xi_0 = \frac{1}{12^6}$. In this case the equation for the j-invariants splits as a product of linear factors

$$j_1 = \frac{u}{u+v},$$
$$j_2 = \frac{v}{u+v}.$$

Now it is possible to see that the threefold $\mathcal{A}$ is birational over $\mathbb{P}^1_{u,v}$ to a fibre product of elliptic surfaces $\mathcal{E}_{1,2} \to \mathbb{P}^1_{u,v}$. Furthermore, these elliptic surfaces are isomorphic, with isomorphism induced by the involution $(u, v) \mapsto (v, u)$ on $\mathbb{P}^1_{u,v}$. Thus it is again enough to just consider $\mathcal{E}_1$.

In this setting, we still find that $\omega_{\mathcal{E}_1} \cong \mathcal{O}_{\mathcal{E}_1}(-E)$, where E denotes the class of a fibre in $\mathcal{E}_1$. This again gives us that the Euler number of $\mathcal{E}_1$ is 12. However, this time we only have three points where $j = 0$, 1, or ∞. In fact, it turns out that in this case $\mathcal{E}_1$ has four singular fibres of types I_1, II, III and I_0^* at $(u, v) = (-1, 1)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ respectively.

Using this we can construct $\mathcal{E}_1$ and, performing the Kummer construction, thus obtain a birational model for W when $\xi_0 = \frac{1}{12^6}$.

References

[CD11] A. Clingher and C. F. Doran, *Note on a geometric isogeny of K3 surfaces*, Int. Math. Res. Not. **2011** (2011), no. 16, 3657–3687.

[CDL+] A. Clingher, C. F. Doran, J. Lewis, A. Y. Novoseltsev, and A. Thompson, *The 14th case VHS via K3 fibrations*, Manuscript in preparation, preprint available December 2012.

[DM06] C. F. Doran and J. W. Morgan, *Mirror symmetry and integral variations of Hodge structure underlying one-parameter families of Calabi-Yau threefolds*, Mirror Symmetry. V, AMS/IP Stud. Adv. Math., vol. 38, Amer. Math. Soc., Providence, RI, 2006, pp. 517–537.