The Arithmetic/Thin Dichotomy for Calabi-Yau Threefolds and Families of K3 Surfaces

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Introduction

Let X be a projective K3 surface and let $\omega \in H^{2,0}(X)$ be a generator.

Recall that the cup-product endows $H^2(X, \mathbb{Z})$ with the structure of a lattice, isometric to the *K3 lattice*

$$\Lambda_{\mathrm{K3}} := H \oplus H \oplus H \oplus (-E_8) \oplus (-E_8)$$

$\omega^{\perp} \cap H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$ is the *Néron-Severi lattice* of X, denoted $\mathrm{NS}(X)$. It is isomorphic to the Picard group $\mathrm{Pic}(X)$.

If N is a lattice of signature $(1, n)$, then X is called *N-polarized* if there exists a primitive lattice embedding $N \hookrightarrow \mathrm{NS}(X)$ such that the image of N contains an ample class.

Aim

Extend these concepts to familes of K3 surfaces.

Families of K3 surfaces

A *family of K3 surfaces* will be a flat surjective morphism $\pi \colon \mathcal{X} \to U$ from a variety $\mathcal{X}$ to a smooth, irreducible, quasiprojective base U, so that for every $p \in U$, the fibre X_p above p is a smooth K3 surface.

Assume also that there is a line bundle $\mathcal{L}$ on $\mathcal{X}$, whose restriction $\mathcal{L}_p$ to X_p is ample and primitive in $\mathrm{Pic}(X_p)$ for each $p \in U$.

Note: We do not assume that U is simply connected!

The relative Néron-Severi group

Next, we define a relative version of the Néron-Severi group.

Using the Hodge filtration $F^\bullet$ on $R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_U$, we may define

$$\mathcal{H}^{2,0}(\mathcal{X}) := F^2(R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_U).$$

Let $\mathcal{NS}(\mathcal{X})$ denote the local subsystem of $R^2\pi_*\mathbb{Z}$ which is orthogonal to a flat local section of $\mathcal{H}^{2,0}(\mathcal{X})$. This is defined globally and equal to the Picard sheaf of π.

Lattice polarized families of K3 surfaces

Let $\mathcal{N}$ be a local subsystem of $\mathcal{NS}(\mathcal{X})$ such that, for any $p \in U$, the fibre $\mathcal{N}_p$ is (non-canonically) isomorphic to a fixed lattice N and is embedded into $H^2(X_p, \mathbb{Z})$ as a primitive sublattice that contains the Chern class of the ample line bundle $\mathcal{L}_p$.

Definition

The family $\mathcal{X}$ is called *N-polarized* if $\mathcal{N}$ is a trivial local system.

Note that a choice of isomorphism $\mathcal{N}_p \cong N$ for any point $p \in U$ determines uniquely an isomorphism $\mathcal{N}_{p'} \cong N$ for every other point $p' \in U$ by parallel transport. So $\mathcal{N}_p$ must be fixed under monodromy in U.

Calabi-Yau threefolds

We can use this machinery to study Calabi-Yau threefolds.

Suppose that $\mathcal{X} \to \mathbb{P}^1$ is a Calabi-Yau threefold fibred by K3 surfaces. Let $U \subset \mathbb{P}^1$ denote the open set over which the fibres of $\mathcal{X}$ are smooth K3 surfaces and let $\mathcal{X}_U \to U$ be the restriction.

From the fibration $\mathcal{X}_U \to U$ we obtain a map $U \to \mathcal{M}_{\mathrm{K3}}$, where $\mathcal{M}_{\mathrm{K3}}$ is the moduli space of K3 surfaces. A lot of the geometry of $\mathcal{X}$ is encoded in this map (think of it as the analogue of the functional invariant map from elliptic surface theory).

However, this map is very hard to study in general: the moduli space $\mathcal{M}_{\mathrm{K3}}$ is very large (20-dimensional) and has unpleasant properties.

Lattice polarized fibrations on Calabi-Yau threefolds

For this reason we turn to Calabi-Yau threefolds fibred by lattice polarized K3 surfaces. Suppose now that $\mathcal{X} \to \mathbb{P}^1$ is a Calabi-Yau threefold fibred by K3 surfaces and that the restriction $\mathcal{X}_U \to U$ to the smooth fibres is an N-polarized family of K3 surfaces.

Then we have a map $U \to \mathcal{M}_N$, the moduli space of N-polarized K3 surfaces.

This moduli space is much better. It is a quasiprojective variety with only quotient singularities and has dimension $20 - \mathrm{rank}(N)$.

The lattices M_n

We focus on the case when the lattice N is equal to

$$M_n := H \oplus (-E_8) \oplus (-E_8) \oplus \langle -2n \rangle,$$

for $1 \leq n \leq 4$.

This lattice has rank 19, so the moduli space $\mathcal{M}_{M_n}$ is a curve.

If $\mathcal{X}_U \to U$ is an M_n-polarized family of K3 surfaces on a Calabi-Yau threefold $\mathcal{X} \to \mathbb{P}^1$, we get a map $U \to \mathcal{M}_{M_n}$ that extends uniquely to a map $\mathbb{P}^1 \to \mathcal{M}_{M_n}$. This map is called the *generalized functional invariant*.

The moduli spaces $\mathcal{M}_{M_n}$

The moduli spaces $\mathcal{M}_{M_n}$ are modular curves

$$\mathcal{M}_{M_n} \cong \Gamma_0(n)^+ \setminus \mathbb{H}.$$

For $1 \leq n \leq 4$, these modular curves contain three orbifold points, of orders

$$\begin{aligned} n = 1: &\quad (2, 3, \infty) \\ n = 2: &\quad (2, 4, \infty) \\ n = 3: &\quad (2, 6, \infty) \\ n = 4: &\quad (2, \infty, \infty) \end{aligned}$$

Why consider these lattices in particular?

Several of our favourite Calabi-Yau threefolds give rise to families of M_n-polarized K3 surfaces. For instance:

- The mirror to the quintic $\mathbb{P}^4[5]$ is fibred by M_2- and M_3-polarized K3 surfaces.
- The mirror to the sextic $\mathbb{WP}(1,1,1,2)[6]$ is fibred by M_1- and M_2-polarized K3 surfaces.
- The mirror to the complete intersection $\mathbb{P}^5[2,4]$ is fibred by M_2-, M_3- and M_4-polarized K3 surfaces.

Generalized functional invariants

All of these Calabi-Yau threefolds are examples of Calabi-Yau threefolds with Hodge number $h^{2,1} = 1$. Many other Calabi-Yau threefolds with $h^{2,1} = 1$ are also fibred by M_n-polarized K3 surfaces.

What do the generalized functional invariants $\mathbb{P}^1 \to \mathcal{M}_{M_n}$ for these threefolds look like?

They all have a common form, determined by a pair of integers (i, j). The map $\mathbb{P}^1 \to \mathcal{M}_{M_n}$ is then an $(i + j)$-fold cover that is ramified totally over the cusp, to orders i and j over the orbifold point of order $\neq 2$, and has one additional simple ramification over a point that is free to vary.

The integers (i, j) are given by the following table.

The integers (i, j)

Lattice	Mirror threefold	Toric?	(i, j)	Arithmetic/thin
M_1	$\mathbb{WP}(1,1,1,1,2)[6]$	Yes	$(1,2)$	Arithmetic
	$\mathbb{WP}(1,1,1,1,4)[8]$	Yes	$(1,3)$	Thin
	$\mathbb{WP}(1,1,1,2,5)[10]$	Yes	$(2,3)$	Arithmetic
	$\mathbb{WP}(1,1,1,1,1,3)[2,6]^*$	Yes	$(1,1)$	Thin
	$\mathbb{WP}(1,1,1,2,2,3)[4,6]^*$	Yes	$(2,2)$	Arithmetic
M_2	$\mathbb{P}^4[5]$	Yes	$(1,4)$	Thin
	$\mathbb{WP}(1,1,1,1,2)[6]$	Yes	$(2,4)$	Arithmetic
	$\mathbb{WP}(1,1,1,1,4)[8]$	Yes	$(4,4)$	Thin
	$\mathbb{P}^5[2,4]$	Yes	$(1,1)$	Thin
	$\mathbb{WP}(1,1,1,1,2,2)[4,4]$	Yes	$(2,2)$	Arithmetic
M_3	$\mathbb{P}^4[5]$	No	$(2,3)$	Thin
	$\mathbb{P}^5[2,4]$	No	$(1,3)$	Thin
	$\mathbb{P}^5[3,3]$	Yes	$(1,2)$	Arithmetic
	$\mathbb{WP}(1,1,1,1,1,2)[3,4]^*$	Yes	$(2,2)$	Arithmetic
	$\mathbb{P}^6[2,2,3]$	Yes	$(1,1)$	Thin
M_4	$\mathbb{P}^5[2,4]$	No	$(2,2)$	Thin
	$\mathbb{P}^6[2,2,3]$	No	$(1,2)$	Thin
	$\mathbb{P}^7[2,2,2,2]$	Yes	$(1,1)$	Thin

Deformations

We expect the form of the generalized functional invariant map to determine much of the geometry of these Calabi-Yau threefolds.

For instance, for a Calabi-Yau threefold $\mathcal{X}$, the value of $h^{2,1}(\mathcal{X})$ is equal to the dimension of the deformation space of $\mathcal{X}$.

Thus the threefolds with $h^{2,1} = 1$ above all move in 1-parameter families. Their deformation spaces may be identified with a copy of $\mathbb{P}^1$ with three points removed (the *thrice-punctured sphere*).

The deformation parameter can be identified with the "free to vary" point in $\mathcal{M}_{M_n}$ over which the generalized functional invariant is simply ramified. The three points that get removed are the three orbifold points.

The arithmetic/thin dichotomy I

Another property of Calabi-Yau threefolds $\mathcal{X}$ with $h^{2,1}(\mathcal{X}) = 1$ is the *arithmetic/thin dichotomy*, discovered by Brav and Thomas.

As already mentioned, these threefolds vary in families over the thrice-punctured sphere. The Hodge decomposition on their third integral cohomology $H^3(\mathcal{X}, \mathbb{Z})$ thus defines an integral variation of Hodge structure over the thrice punctured sphere.

The possible integral variations of Hodge structure that can arise this way were classified by Doran and Morgan. They found 14 cases. Twelve of these cases are realized by threefolds from our table.

The arithmetic/thin dichotomy II

Monodromy around the punctures in the thrice-punctured sphere acts on $H^3(\mathcal{X}, \mathbb{Z})$. This monodromy action defines a Zariski dense subgroup of $\mathrm{Sp}(4, \mathbb{R})$, which may be either arithmetic or non-arithmetic (more commonly called *thin*).

Brav, Singh, Thomas, and Venkataramana have studied this action in each of Doran and Morgan's 14 cases. They found that seven cases are arithmetic and seven cases are thin.

It is an open question to explain this dichotomy in terms of the geometry of the Calabi-Yau threefolds.

An observation

From our table, we can make the following observation:

Theorem

Suppose that $\mathcal{X}$ is a Calabi-Yau threefold from our table that admits a torically-induced fibration by M_n-polarized K3 surfaces. Then $\mathcal{X}$ has thin monodromy if and only if neither i nor j is equal to 2.

This suggests that the generalized functional invariant map has some control over the arithmetic/thin dichotomy.

Extensions

So far we have constrained ourselves to consider only Calabi-Yau threefolds $\mathcal{X}$ with $h^{2,1}(\mathcal{X}) = 1$. However, other Calabi-Yau threefolds can admit fibrations $\mathcal{X} \to \mathbb{P}^1$ by M_n-polarized K3 surfaces too.

In all such examples, we again have a generalized functional invariant map $\mathbb{P}^1 \to \mathcal{M}_{M_n}$. What forms do these maps take? And how do they control the geometry of the Calabi-Yau threefold?

Observations about the generalized functional invariant

We first observe that the Calabi-Yau condition for the threefolds places strict constraints on the form that the generalized functional invariant map can take. For instance:

- The degree of the cover $\mathbb{P}^1 \to \mathcal{M}_{M_n}$ is constrained to take only certain values.
- The ramification of the generalized functional invariant map over the orbifold point of order $\neq 2$ in $\mathcal{M}_{M_n}$ is also tightly constrained.
- Except in some special cases, the generalized functional invariant map cannot be ramified over the 2-orbifold point in $\mathcal{M}_{M_n}$.

Observations about geometry

We also observe that ramification of the generalized functional invariant away from the orbifold points in $\mathcal{M}_{M_n}$ seems to be related to the Hodge number $h^{2,1}$ of the Calabi-Yau threefold.

This suggests that there is a link between spaces of deformations of the generalized functional invariant map and the moduli spaces of Calabi-Yau threefolds fibred by M_n-polarized K3 surfaces.