Pseudolattices, degenerations, and fibrations of K3 surfaces

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Aim

Develop a lattice theory for Type II degenerations that is natural for mirror symmetry.

1 - Pseudolattices

Pseudolattices

Definition (Kuznetsov 2017)

A *pseudolattice* is a finitely generated free abelian group G equipped with a (not necessarily symmetric) nondegenerate bilinear form

$$\langle \cdot, \cdot \rangle_{\mathrm{G}} \colon \mathrm{G} \times \mathrm{G} \to \mathbb{Z}.$$

Definition (Kuznetsov 2017)

A *Serre operator* S_{G} is an automorphism of G so that

$$\langle u, v \rangle_{\mathrm{G}} = \langle v, S_{\mathrm{G}}(u) \rangle_{\mathrm{G}} \qquad \forall u, v \in \mathrm{G}$$

Surface-like pseudolattices

Definition (Kuznetsov 2017)

A pseudolattice G is *surface-like* if there is a primitive element $p \in \mathrm{G}$ such that

1. $\langle p, p \rangle_{\mathrm{G}} = 0$.
2. $\langle p, u \rangle_{\mathrm{G}} = \langle u, p \rangle_{\mathrm{G}}$ for all $u \in \mathrm{G}$.
3. If $u, v \in \mathrm{G}$ satisfy $\langle u, p \rangle_{\mathrm{G}} = \langle v, p \rangle_{\mathrm{G}} = 0$, then $\langle u, v \rangle_{\mathrm{G}} = \langle v, u \rangle_{\mathrm{G}}$.

An element p with the above properties is called a *point-like* element in G.

Definition (Kuznetsov 2017)

The *Néron-Severi lattice* of a surface-like pseudolattice G is the lattice

$$\mathrm{NS}(\mathrm{G}) := p^{\perp}/p.$$

$\mathrm{NS}(\mathrm{G})$ contains a special element K_{G}; the *canonical class* of G.

Example

Example

Let X be a (smooth complex projective) surface.

- $\mathrm{G} = \mathrm{K}_0^{\mathrm{num}}(\mathbf{D}(X))$ equipped with the Euler pairing is surface-like with point-like element $[\mathcal{O}_{\mathrm{pt}}]$.
- G has Serre operator $S_{\mathrm{G}} = (- \otimes \omega_X)[n]$.
- $\mathrm{NS}(\mathrm{G}) = \mathrm{NS}(X)$.
- $K_{\mathrm{G}} = [\omega_X]$.

Spherical homomorphisms

Definition (Harder-T. 2020)

A *spherical homomorphism* is a homomorphism of abelian groups $f\colon \mathrm{G} \to \mathrm{H}$ with the following properties:

1. The homomorphism f has a right adjoint $r\colon \mathrm{H} \to \mathrm{G}$, in the sense that

$$\langle f(u), v\rangle_{\mathrm{H}} = \langle u, r(v)\rangle_{\mathrm{G}}$$

   for any $u \in \mathrm{G}$ and $v \in \mathrm{H}$.

2. The *twist* and *cotwist* endomorphisms, which are defined as

$$T_f := \mathrm{id}_{\mathrm{H}} - fr, \qquad C_f := \mathrm{id}_{\mathrm{G}} - rf,$$

   respectively, are invertible.

$f\colon \mathrm{G} \to \mathrm{H}$ is *relative* $(-1)^n$*-CY* if G has a Serre operator S_{G} and $C_f = (-1)^n S_{\mathrm{G}}$.

Example

Example

Let X be a surface such that $|-K_X|$ contains a smooth genus 1 curve C and let $i\colon C \hookrightarrow X$ denote the inclusion.

Let $\mathrm{E} := \mathrm{K}_0^{\mathrm{num}}(\mathbf{D}(C))$ (equipped with the Euler pairing) denote the *elliptic curve pseudolattice*.

Then $i^*\colon \mathrm{K}_0^{\mathrm{num}}(\mathbf{D}(X)) \to \mathrm{E}$ is a relative $(-1)^0$-CY spherical homomorphism with right adjoint i_*.

We call a spherical homomorphism $\mathrm{G} \to \mathrm{E}$ *quasi del Pezzo* if it "looks like" this example.

Gluing pseudolattices along spherical homomorphisms

Definition (Harder-T. 2020)

Assume that $f_i \colon \mathrm{G}_i \longrightarrow \mathrm{E}$ are spherical homomorphisms for $i \in \{1, 2\}$. Define $\mathrm{G}_1 \oplus_{\mathrm{E}} \mathrm{G}_2$ to be the pseudolattice with underlying group $\mathrm{G}_1 \oplus \mathrm{G}_2$ and bilinear form $\langle \cdot, \cdot \rangle_{\mathrm{G}_1 \oplus_{\mathrm{E}} \mathrm{G}_2}$ given as follows: for $u_i \in \mathrm{G}_i$ and $v_j \in \mathrm{G}_j$, define

$$\langle u_i, v_j \rangle_{\mathrm{G}_1 \oplus_{\mathrm{E}} \mathrm{G}_2} = \begin{cases} \langle u_i, v_j \rangle_{\mathrm{G}_i} & \text{if } i = j \\ \langle f_i(u_i), f_j(v_j) \rangle_{\mathrm{E}} & \text{if } i = 1,\ j = 2 \\ 0 & \text{if } i = 2,\ j = 1 \end{cases}$$

and extend to $\mathrm{G}_1 \oplus \mathrm{G}_2$ by linearity.

Gluing quasi del Pezzo homomorphisms

Let $f_i \colon \mathrm{G}_i \longrightarrow \mathrm{E}$ be quasi del Pezzo homomorphisms for $i \in \{1, 2\}$.

Assume that $\langle K_{\mathrm{G}_1}, K_{\mathrm{G}_1} \rangle_{\mathrm{G}_1} = -\langle K_{\mathrm{G}_2}, K_{\mathrm{G}_2} \rangle_{\mathrm{G}_2}$.

Let $\mathrm{G} := \mathrm{G}_1 \oplus_{\mathrm{E}} \mathrm{G}_2$ glued along f_1 and $(-f_2)$. Then $f := f_1 \oplus (-f_2)$ is a relative $(-1)^0$-Calabi-Yau spherical homomorphism. Let r be its right adjoint.

An exact sequence of lattices

Lemma

Let $\mathrm{K} = \ker(f)$ and $\overline{\mathrm{E}}$ be the saturation of $r(\mathrm{E})$ in G. Then

- The bilinear form is symmetric on K, so K is a lattice.
- $\overline{\mathrm{E}} \subset \mathrm{K}$ is a totally degenerate sublattice.

Let $\mathrm{M} = \mathrm{K}/\overline{\mathrm{E}}$. Then we have an exact sequence of lattices

$$0 \longrightarrow \overline{\mathrm{E}} \longrightarrow \mathrm{K} \longrightarrow \mathrm{M} \longrightarrow 0.$$

2 - Tyurin degenerations of K3 surfaces

Tyurin degenerations

Let $\mathcal{V} \to \Delta$ denote a semistable Type II degeneration of K3 surfaces over the open unit disc in $\mathbb{C}$, with general fibre $V \subset \mathcal{V}$ and special fibre $V_0 \subset \mathcal{V}$.

We assume that $\pi \colon \mathcal{V} \to \Delta$ is a *Tyurin degeneration*, meaning that $V_0 := X_1 \cup_C X_2$ consists of a pair of smooth rational surfaces X_i glued along a smooth anticanonical divisor C.

Let $\mathrm{G}_i := \mathrm{K}_0^{\mathrm{num}}(\mathbf{D}(X_i))$ and let $f_i \colon \mathrm{G}_i \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{D}(C)) = \mathrm{E}$ be the pull-backs by the inclusions.

A geometric interpretation for G and K

Lemma

The Chern character map induces a natural isomorphism of $\mathbb{Q}$-modules

$$\mathrm{G}_{\mathbb{Q}} \cong H^{\mathrm{even}}(X_1, \mathbb{Q}) \oplus H^{\mathrm{even}}(X_2, \mathbb{Q}),$$

where $\mathrm{G}_{\mathbb{Q}} := \mathrm{G} \otimes \mathbb{Q}$. The restriction to $\mathrm{K}_{\mathbb{Q}} := \mathrm{K} \otimes \mathbb{Q}$ induces an isomorphism

$$\mathrm{K}_{\mathbb{Q}} \cong H^0(V_0, \mathbb{Q}) \oplus \mathrm{Gr}_2^W H^2(V_0, \mathbb{Q}) \oplus H^4(V_0, \mathbb{Q}),$$

where $W_\bullet$ is the weight filtration arising from Deligne's mixed Hodge structure on a normal crossing variety.

A geometric interpretation for M

Lemma

There is an isomorphism of lattices

$$M \cong H^0_{\mathrm{lim}}(V) \oplus \mathrm{Gr}^W_2 H^2_{\mathrm{lim}}(V) \oplus H^4_{\mathrm{lim}}(V),$$

where $W_\bullet$ is the weight filtration arising from the limiting mixed Hodge structure (which is defined over $\mathbb{Z}$) and the right-hand side is equipped with the bilinear form induced from the Mukai lattice.

A geometric interpretation of the exact sequence

Proposition

There is a commutative diagram of $\mathbb{Q}$-modules

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\overline{\mathrm{E}}_{\mathbb{Q}} & \xrightarrow{\cong} & \mathbb{Q}\xi \oplus \mathbb{Q}\eta \\
\downarrow & & \downarrow \\
\mathrm{K}_{\mathbb{Q}} & \xrightarrow{\cong} & H^0(V_0) \oplus \mathrm{Gr}_2^W H^2(V_0) \oplus H^4(V_0) \\
\downarrow & & \downarrow \\
\mathrm{M}_{\mathbb{Q}} & \xrightarrow{\cong} & H^0_{\mathrm{lim}}(V) \oplus \mathrm{Gr}_2^W H^2_{\mathrm{lim}}(V) \oplus H^4_{\mathrm{lim}}(V) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where

$\xi = (-C, C) \in H^2(X_1) \oplus H^2(X_2), \qquad \eta = (-\mathrm{pt}, \mathrm{pt}) \in H^4(X_1) \oplus H^4(X_2).$

Integrality

The exact sequence on the right is a weight graded piece of Clemens-Schmid. Friedman showed this is exact over $\mathbb{Z}$ in 1984.

The isomorphisms on the previous slide do not hold over $\mathbb{Z}$, but the failure is confined to H^4. Restriction to Néron-Severi lattices gives the exact sequence over $\mathbb{Z}$:

$$0 \longrightarrow \mathbb{Z}\xi \longrightarrow \mathrm{Gr}_2^W H^2(V_0, \mathbb{Z}) \longrightarrow \mathrm{Gr}_2^W H^2_{\mathrm{lim}}(V, \mathbb{Z}) \longrightarrow 0.$$

3 - Elliptic fibrations on K3 surfaces

Elliptic fibrations on K3 surfaces

Let $\pi\colon Y \to \mathbb{P}^1$ be a K3 surface fibred by genus 1 curves (not necessarily with a section).

Pick a loop $\gamma \subset \mathbb{P}^1$ dividing $\mathbb{P}^1 = \Delta_1 \cup_\gamma \Delta_2$ into two closed discs and let $\pi_i\colon Y_i \to \Delta_i$ denote the restrictions. Assume that the fibres over γ are all smooth. Let $p \in \gamma$ be a point and F be the fibre over p.

There is a bilinear form called the *Seifert pairing* making $H_2(Y_i, F; \mathbb{Z})$ into a pseudolattice and a spherical homomorphism $\phi_i\colon H_2(Y_i, f; \mathbb{Z}) \to H_1(F, \mathbb{Z})$, where $H_1(F, \mathbb{Z}) \cong \mathrm{E}$ is equipped with the standard symplectic pairing.

Pseudolattices from elliptic fibrations

Assume that there exists a symplectic basis for $H^1(F, \mathbb{Z})$ such that for both $i \in \{1, 2\}$:

1. Anticanonical monodromy around the boundary of Δ_i acts as

$$\begin{pmatrix} 1 & e(Y_i) - 12 \\ 0 & 1 \end{pmatrix},$$

   where $e(Y_i)$ is the topological Euler number of Y_i.
2. If r_i is the right adjoint to ϕ_i, then $r(1,0)$ is primitive.

Theorem

Under these assumptions, ϕ_1 and ϕ_2 are quasi del Pezzo homomorphisms of pseudolattices.

Pseudolattices from elliptic fibrations

Let $\mathrm{G}_i = H_2(Y_i, F; \mathbb{Z})$ and define G, K, M as before.

Proposition

There are isomorphisms of lattices

$$\mathrm{K} \cong H^2_c(Y \backslash F, \mathbb{Z})/\mathbb{Z}F,$$
$$\mathrm{M} \cong F^\perp/\mathbb{Z}F \subset H^2(Y, \mathbb{Z}),$$

and a commutative diagram of $\mathbb{Z}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\mathrm{E}} & \longrightarrow & \mathrm{K} & \longrightarrow & \mathrm{M} & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \\ 0 & \longrightarrow & H^1(F; \mathbb{Z}) & \longrightarrow & H^2_c(U; \mathbb{Z})/\mathbb{Z}F & \longrightarrow & F^\perp/\mathbb{Z}F & \longrightarrow & 0 \end{array}$$

Remarks

The bottom sequence is a perverse graded piece of the "mirror Clemens-Schmid" sequence [Doran-T. 2021].

Restriction to Néron-Severi lattices does not give a particularly meaningful sequence on this side; it is better to work on the level of the lattices K and M.

4 - Mirror symmetry

A conjecture

Conjecture (Doran-Harder-T. 2017)

If X and $\check{X}$ are a mirror pair of K3 surfaces, there should be a correspondence

$$\text{Type II degenerations of } X \longleftrightarrow \text{Genus 1 fibrations on } \check{X}$$
$$\text{Kulikov models} \longleftrightarrow \text{Splittings of } \mathbb{P}^1$$
$$\text{Components} \longleftrightarrow \text{LG models}$$

Outlook

The pseudolattice picture presented here gives an "ambient lattice theory" for degenerations and fibrations, which is identical on both sides of the correspondence above.

Our ongoing work seeks to understand the interaction between this and the theory of lattice polarisations on K3 surfaces, in order to prove the conjecture on the previous slide.

Thanks for your attention!