Anticanonical Pairs with Involution

Alan Thompson (joint work with Valery Alexeev)

University of Warwick / University of Cambridge

Setup

We study *anticanonical pairs with involution*. These are triples (X, D, ι), where

- X is a normal rational surface,
- D is an effective reduced divisor on X with $K_X + D \sim 0$,
- $\iota\colon X \to X$ is an involution such that $\iota(D) = D$,

under the assumptions that

(a) (positivity) the ramification divisor R of ι is Cartier and ample, and

(b) (singularity) the pair $(X, D + \epsilon R)$ has log canonical singularities for $0 < \epsilon \ll 1$.

For simplicity of presentation, we also assume

(c) (Type III) if $f\colon \widetilde{X} \to X$ is a minimal resolution of the singularities of X and $\widetilde{D}$ is the divisor on $\widetilde{X}$ defined by $K_{\widetilde{X}} + \widetilde{D} \sim f^*(K_X + D) \sim 0$, then $\widetilde{D}$ is a cycle of $\mathbb{P}^1$'s.

Let $\pi\colon X \to X/\iota := Y$ be the quotient, set $C = \pi(D)$ to be the boundary on Y, and $B = \pi(R)$ to be the branch divisor. Then study of (X, D, ι) is equivalent to study of (Y, C, B).

Toric Examples

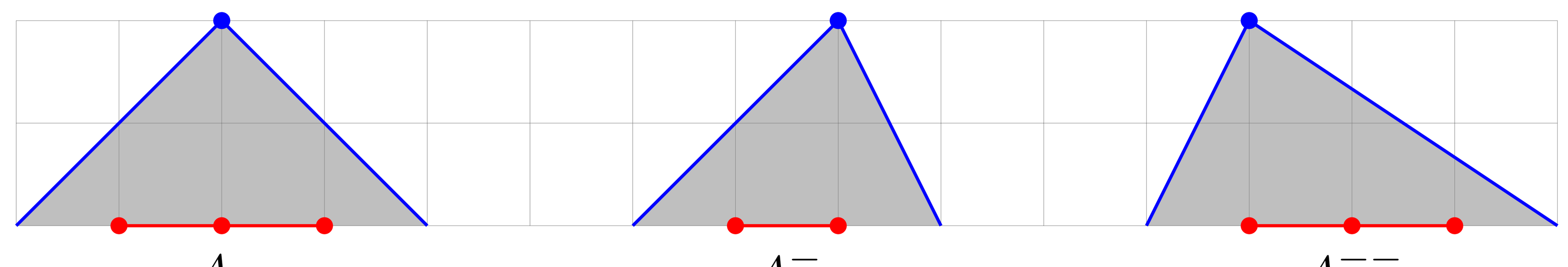
A polytope P with integral vertices corresponds to a polarised toric variety (Y, L_P), with L_P an ample line bundle. In the panel to the right, we list polytopes P giving rise to toric examples of triples (Y, C, B) as above; in each case we list the vertices of P and draw a representative example. The divisor C is part of the toric boundary and has two torus-invariant components, corresponding to the two blue sides passing through the blue vertex $v = (2, 2)$, and $\mathcal{O}_Y(B) \cong L_P$. Finally, we label each case by a Dynkin diagram, shown in red.

Toric Polytopes

(1) A_{2n-1}: $(2,2)$, $(0,0)$, $(2n,0)$.

(2) A^-_{2n-2}: $(2,2)$, $(0,0)$, $(2n-1,0)$.

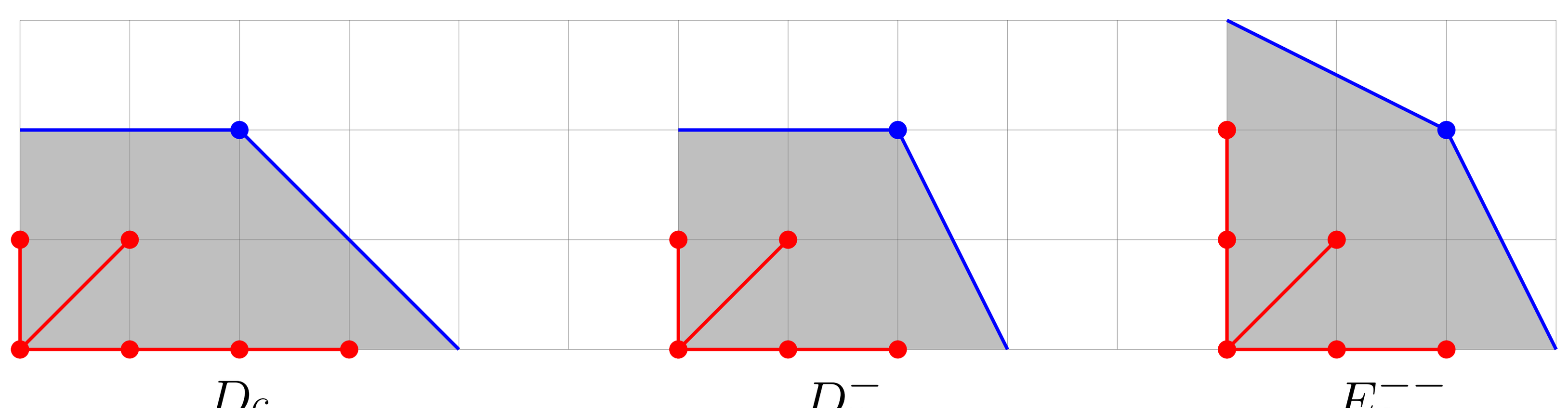
(3) A^{--}_{2n-3}: $(2,2)$, $(1,0)$, $(2n-1,0)$.



(4) D_{2n}: $(2,2)$, $(0,2)$, $(0,0)$, $(2n-2,0)$.

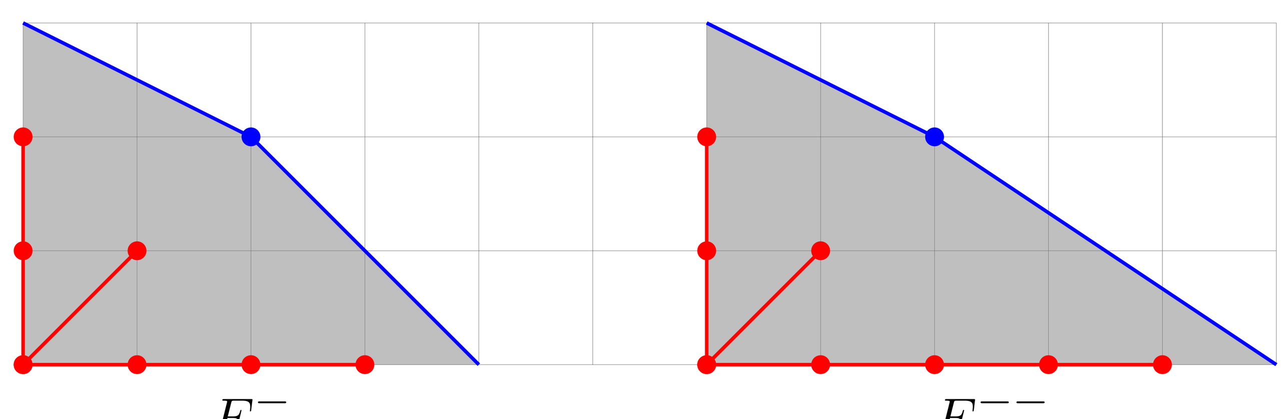
(5) D^-_{2n-1}: $(2,2)$, $(0,2)$, $(0,0)$, $(2n-3,0)$.

(6) E_6^{--}: $(2,2)$, $(0,3)$, $(0,0)$, $(3,0)$.



(7) E_7^-: $(2,2)$, $(0,3)$, $(0,0)$, $(4,0)$.

(8) E_8^{--}: $(2,2)$, $(0,3)$, $(0,0)$, $(5,0)$.



Main Result

Theorem. *Every Type III anticanonical pair with involution (X, D, ι) is either*

- *(pure type) a double cover of one of the toric examples (Y, B, C) from the panel on the left, or*
- *(primed type) a blow-up of a pure type at one or more of the points $D \cap R$.*

Moreover, the moduli space of pure type anticanonical pairs with involution may be identified with the quotient $\mathrm{Hom}(\Lambda, \mathbb{C}^*)/W_\Lambda$, where Λ is the root lattice associated to the corresponding Dynkin diagram and W_Λ is its Weyl group.

Moduli and Losev-Manin Spaces

The space $\mathrm{Hom}(\Lambda, \mathbb{C}^*)$ admits a natural compactification to a toric variety $X(\Lambda)$ and the action of W_Λ extends. Moreover, the boundary points in $X(\Lambda)/W_\Lambda$ provide moduli for degenerate anticanonical pairs with involution in a natural way.

In 2000, Losev and Manin showed that $X(A_n)$ may be realised as a compact moduli space for stable $(n+1)$-pointed chains of $\mathbb{P}^1$'s. One may show that there is a natural correspondence between anticanonical pairs with involution that have Dynkin diagram A_n and configurations of $(n+1)$ points in $\mathbb{P}^1$. This correspondence identifies the moduli space of A_n-type anticanonical pairs with involution with a Losev-Manin moduli space.

One may therefore think of the moduli spaces of D_n- and E_n-type anticanonical pairs with involution as generalisations of Losev-Manin spaces to other simply-laced Dynkin diagrams.