Families of lattice polarized K3 surfaces with monodromy

Alan Thompson

Fields Institute / University of Alberta / University of Waterloo

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(Joint work with C. Doran, A. Harder and A. Novoseltsev)

Introduction

Let X be a K3 surface and let $\omega \in H^{2,0}(X)$ be a generator.

Recall that the cup-product endows $H^2(X,\mathbb{Z})$ with the structure of a lattice, isometric to the *K3 lattice*

$$\Lambda_{\mathrm{K3}} := H \oplus H \oplus H \oplus (-E_8) \oplus (-E_8)$$

$\omega^\perp \cap H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{C})$ is the *Néron-Severi group* of X, denoted $\mathrm{NS}(X)$.

If N is a lattice of signature $(1,n)$, then X is called *N-polarized* if there exists a primitive lattice embedding $N \hookrightarrow \mathrm{NS}(X)$.

Aim

Extend this definition to families of K3 surfaces.

Definitions

A *family of K3 surfaces* will be a flat surjective morphism $\pi\colon \mathcal{X} \to U$ from a variety $\mathcal{X}$ to a smooth, irreducible, quasiprojective base U, so that for every $p \in U$, the fibre X_p above p is a smooth K3 surface.

Assume also that there is a line bundle $\mathcal{L}$ on $\mathcal{X}$, whose restriction $\mathcal{L}_p$ to X_p is ample and primitive in $\mathrm{Pic}(X_p)$ for each $p \in U$.

Note: We do not assume that U is simply connected!

The simple answer

Hosono, Lian, Oguiso and Yau give a simple answer to the problem of extending a lattice polarization to such a family:

Definition

A family K3 surfaces is called *N-polarizable* if each fibre is N-polarized.

This definition does not provide any control over the embeddings $N \hookrightarrow \mathrm{NS}(X_p)$, which are only fixed up to automorphisms of the K3 lattice.

We would like to study the action of monodromy on classes in $\mathrm{NS}(X_p)$, so this definition is too weak.

More definitions

To do better, we begin by defining a relative version of the Néron-Severi group.

Using the Hodge filtration $F^\bullet$ on $R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_U$, we may define

$$\mathcal{H}^{2,0}(\mathcal{X}) := F^2(R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_U).$$

Let $\mathcal{NS}(\mathcal{X})$ denote the local subsystem of $R^2\pi_*\mathbb{Z}$ which is orthogonal to a flat local section of $\mathcal{H}^{2,0}(\mathcal{X})$. This is defined globally and equal to the Picard sheaf of π.

Another answer

Let $\mathcal{N}$ be a local subsystem of $\mathcal{NS}(\mathcal{X})$ such that, for any $p \in U$, the fibre $\mathcal{N}_p$ embeds into $H^2(X_p, \mathbb{Z})$ as a primitive sublattice that is (non-canonically) isomorphic to a fixed lattice N.

Definition

The family $\mathcal{X}$ is called *N-polarized* if $\mathcal{N}$ is a trivial local system.

This is too constrained: a choice of isomorphism $\mathcal{N}_p \cong N$ for any point $p \in U$ determines uniquely an isomorphism $\mathcal{N}_{p'} \cong N$ for every other point $p' \in U$ by parallel transport. So monodromy must act trivially on $\mathcal{N}_p$.

Aim

We want to find something in between the underconstrained definition of *N-polarizable*, where we have no control over the action of monodromy, and the overconstrained definition of *N-polarized*, where the action of monodromy is forced to be trivial.

Monodromy representations

Choose a base point $p \in U$. Parallel transport around loops in U gives a monodromy representation

$$\rho_{\mathcal{X}} \colon \pi_1(U, p) \longrightarrow \mathrm{O}(H^2(X, \mathbb{Z}))$$

which restricts to a monodromy representation

$$\rho_{\mathcal{NS}} \colon \pi_1(U, p) \longrightarrow \mathrm{O}(\mathrm{NS}(X_p)).$$

Similarly, for any local subsystem $\mathcal{N}$ of $R^2\pi_*\mathbb{Z}$ we get a monodromy representation $\rho_{\mathcal{N}}$.

Note: If $\mathcal{X}$ is $\mathcal{N}$-polarized, the image of $\rho_{\mathcal{N}}$ will be the trivial subgroup.

Final answer

Let N^* be the dual lattice of N. Define the *discriminant lattice* of N to be $A_N := N^*/N$; there is a natural map $\alpha_N \colon \mathrm{O}(N) \to \mathrm{Aut}(A_N)$.

Definition

Let N be an even lattice of signature $(1, n)$ and let G be a subgroup of $\mathrm{Aut}(A_N)$. Assume that there is a local subsystem $\mathcal{N} \subset \mathcal{NS}(\mathcal{X})$ with fibres isometric to N and primitively embedded into $H^2(X_p, \mathbb{Z})$. Then $\mathcal{X} \to U$ is called an (N, G)*-polarized family of K3 surfaces* if the restriction of α_N to the image of $\rho_{\mathcal{N}}$ is injective and has image inside G.

Note: Let Id denote the trivial subgroup of $\mathrm{Aut}(A_N)$. Then a family is N-polarized if and only if it is (N, Id)-polarized.

Applications

Theorem

Let $\mathcal{X} \to U$ be a family of K3 surfaces with generic Néron-Severi group N. Then $\mathcal{X} \to U$ is (N, G)-polarized for some $G \subset \mathrm{Aut}(A_N)$ if and only if there is no $\gamma \in \pi_1(U, p)$ such that $\rho_{\mathcal{NS}}(\gamma) = \sigma|_{\mathrm{NS}(X_p)}$, where σ is a symplectic automorphism of X_p.

Corollary

Any family of K3 surfaces with generic Néron-Severi group N having $\mathrm{rank}(N) < 9$ is (N, G)-polarized for some $G \subset \mathrm{Aut}(N)$.

Applications 2

Theorem

Let $\mathcal{X} \to U$ be a family of K3 surfaces with generic Néron-Severi group N. Suppose that a general fibre of $\mathcal{X}$ admits a symplectic automorphism σ. Then σ extends to an automorphism of $\mathcal{X}$ if and only if its action on $\mathrm{NS}(X_p)$ commutes with the image of $\rho_{\mathcal{X}}$.

Corollary

If $\mathcal{X} \to U$ is N-polarized, then σ extends to an automorphism of $\mathcal{X}$.

Shioda-Inose structures

Suppose now that $\mathcal{X} \to U$ is an N-polarized family of K3 surfaces with generic Néron-Severi group N and suppose that there exists an embedding of $E_8 \oplus E_8$ into N.

This defines a *Shioda-Inose structure* on a general fibre X_p of $\mathcal{X}$, giving a canonical Nikulin involution β on X_p. The resolved quotient $\widetilde{X_p/\beta}$ is a Kummer surface.

By the previous corollary, this involution extends to a global involution on $\mathcal{X}$. Let $\mathcal{Y}$ be the resolved quotient. Then $\mathcal{Y}$ is fibred by Kummer surfaces.

A question

Question: Can we find a family of Abelian surfaces $\mathcal{A}$ that give $\mathcal{Y}$ under fibrewise application of the Kummer construction?

Answer: Yes, if $\mathcal{Y}$ is $N' := \mathrm{NS}(Y_p)$-polarized.

We call this process *undoing the Kummer construction*.

However...

In general this does not hold. Instead, $\mathcal{Y}$ is (N', G)-polarized for

$$G = \ker(\alpha_{N^\perp})/\ker(\alpha_{N^\perp(2)}).$$

Thus if U is one dimensional, then we can find a $|G|$-sheeted cover $f\colon U' \to U$ so that the pull-back $f^*\mathcal{Y}$ is N'-polarized. This allows us to undo the Kummer construction for $f^*\mathcal{Y}$.

Calabi-Yau threefolds

Many of our favourite Calabi-Yau threefolds admit M_n-polarized K3 fibrations $\mathcal{X} \to U$, where M_n is the lattice

$$M_n := H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle.$$

For instance, the quintic mirror admits an M_2-polarized K3 fibration and the mirror to the sextic $\mathbb{WP}(1,1,1,1,2)[6]$ admits an M_1-polarized K3 fibration.

The K3 fibres in these fibrations admit Shioda-Inose structures, which define global fibrewise Nikulin involutions as before.

The groups G

The resolved quotient threefolds are new Calabi-Yau threefolds $\mathcal{Y} \to U$, fibred by Kummer surfaces. To study them, we would like to construct them directly by the fibrewise Kummer construction. To do this, we need to undo the Kummer construction for the threefolds $\mathcal{Y}$.

We begin by computing the groups G. For instance:

- In the M_1-polarized case, we have $G = S_3 \times C_2$.
- In the M_2-polarized case, we have $G = D_8$.

Generalized functional invariants

To go further we begin by noting that, in each case, the K3 fibration $\mathcal{X} \to U$ on the Calabi-Yau threefold is the pull-back of a K3 fibration $\mathcal{X}_n$ from the moduli space $\mathcal{M}_{M_n}$ of M_n-polarized K3 surfaces, via the *generalized functional invariant map* Φ:

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}_n \\ \downarrow & & \downarrow \\ U & \xrightarrow{\Phi} & \mathcal{M}_{M_n} \end{array}$$

Covers

If we quotient the threefolds $\mathcal{X}_n$ by the fibrewise Nikulin involution and resolve, we obtain new threefolds $\mathcal{Y}_n \to \mathcal{M}_{M_n}$ that are fibred by Kummer surfaces.

The Calabi-Yau threefolds $\mathcal{Y}$ are birational to the pull-backs $\Phi^*\mathcal{Y}_n$ by the generalized functional invariant map.

If we can find a cover $f\colon C_{M_n} \to \mathcal{M}_{M_n}$ so that the pull-back $f^*\mathcal{Y}_n$ is N'-polarized (where N' is the Néron-Severi lattice of a generic fibre of $\mathcal{Y}_n$), then we can undo the Kummer construction for $f^*\mathcal{Y}_n$.

Finally, we can then undo the Kummer construction for $\mathcal{Y}$ by pulling-back to the fibre product

$$\begin{array}{ccc} U \times_{\mathcal{M}_{M_n}} C_{M_n} & \longrightarrow & U \\ \downarrow & & \downarrow \Phi \\ C_{M_n} & \xrightarrow{\ f\ } & \mathcal{M}_{M_n} \end{array}$$

Modular curves

What are the covers C_{M_n}?

In each case, $\mathcal{M}_{M_n}$ is a modular curve and C_{M_n} is a $|G|$-fold cover of it.

For instance, for $n = 2$ (corresponding to the quintic mirror, amongst other examples), we have

$$\mathcal{M}_{M_2} \cong \Gamma_0(2)^+ \setminus \mathbb{H}.$$

Then C_{M_2} is the cover

$$C_{M_2} \cong (\Gamma_0(4) \cap \Gamma(2)) \setminus \mathbb{H} \longrightarrow \Gamma_0(2)^+ \setminus \mathbb{H}.$$

This is an 8-fold cover, as expected from $G = D_8$.