Degenerations of plane sextics

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Motivation

Our eventual goal is to study the moduli space of K3 surfaces of degree two, i.e. K3 surfaces containing an ample divisor that has self-intersection number 2.

Any K3 surface X of degree two may be realised as a double cover $X \to \mathbb{P}^2$, branched along a smooth sextic curve.

Therefore, to study the moduli space of K3 surfaces of degree two, it is instructive to study the moduli of pairs $(\mathbb{P}^2, B)$, where $B \subset \mathbb{P}^2$ is a smooth sextic curve.

Compactification

Hacking tells us that the moduli space of pairs $(\mathbb{P}^2, B)$ may be compactified to a moduli space of *semistable pairs*.

Definition

A pair (X, B) is called *semistable* if

- X is a normal, log terminal surface and B is a divisor on X,
- the pair $(X, \frac{1}{2}B)$ is log canonical,
- $2K_X + B$ is linearly equivalent to 0, and
- there is a deformation of (X, B) with general fibre a smooth sextic in $\mathbb{P}^2$.

Note: The moduli space of semistable pairs is not separated. We will return to this problem later.

The boundary of this moduli space

Our first aim is to study the boundary of this moduli space.

A generic point in a boundary component corresponds to a pair (X, B) where every singularity of $(X, \frac{1}{2}B)$ is *maximally* log canonical.

Therefore, to study components of the boundary, we need to study pairs (X, B) whose singularities are all maximally log canonical.

A simple case

To simplify matters, consider the case where the normal, log terminal surface X in our semistable pair is isomorphic to $\mathbb{P}^2$.

In this case, semistability implies that B is a sextic curve. Any singularities of the pair $(\mathbb{P}^2, \frac{1}{2}B)$ must arise from singularities of B.

The singularities of B that give rise to maximally log canonical singularities of $(\mathbb{P}^2, \frac{1}{2}B)$ can be easily deduced from results of Kollár, Shepherd-Barron and Alexeev.

As a starting point towards a more general result, we might hope to classify sextics having such singularities.

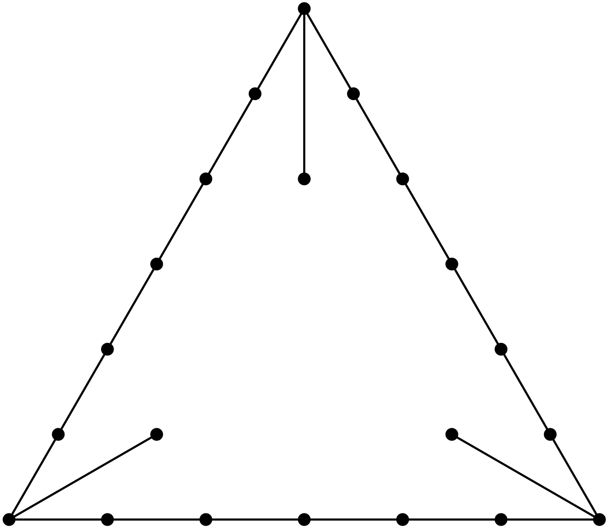
Digression: Coxeter diagrams

Our classification will be in terms of certain Coxeter diagrams. We identify three classes of Coxeter diagrams:

- A diagram is called *elliptic* if it is a Dynkin diagram A_n, D_n, E_n.
- A diagram is called *parabolic* if it is an extended Dynkin diagram $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_n$.
- A diagram is called *hyperbolic* if it is not one of the types above.

The Γ_{21} diagram

We call the following hyperbolic Coxeter diagram Γ_{21}:



Classification theorem

We can now state our first main theorem:

Theorem

Let $(\mathbb{P}^2, B)$ be a semistable pair. Assume that B is singular and that all singularities of $(\mathbb{P}^2, \frac{1}{2}B)$ are maximally log canonical. There is a bijective correspondence between equisingular deformation classes of such pairs and subdiagrams $\Gamma \subset \Gamma_{21}$, up to the action of S_3, such that:

- *every connected component of Γ is either parabolic or hyperbolic;*
- *degeneration of pairs corresponds to inclusion of subdiagrams, with larger subdiagrams being more degenerate.*

Remark: The parabolic/hyperbolic distinction in this theorem corresponds precisely with the Type II/Type III distinction for degenerate K3 surfaces.

A problem

Next, one might try to study the case of more general normal, log terminal surfaces X.

However, things quickly get intractable. This is related to the fact that the moduli space of semistable pairs is not separated, meaning that, in general, a family of semistable pairs over a punctured disc may have many distinct completions.

A solution

To solve this problem, we follow Hacking and instead consider the moduli space of *stable pairs*. This is another compactification of the moduli space of pairs $(\mathbb{P}^2, B)$.

The definition of a stable pair is similar to a semistable pair, except:

- We tighten the condition on singularities of B, requiring $(X, (\frac{1}{2} + \epsilon)B)$ to be semi log canonical for some $\epsilon > 0$.
- However, in return, we need to allow the surface X to be non-normal.

This moduli space is much better behaved: given a family of sextics in $\mathbb{P}^2$ over the punctured disc, there is a unique way to complete it to a family of stable pairs.

Another problem

However, the non-normality of the surface X here is a major obstacle to understanding the boundary of this moduli space.

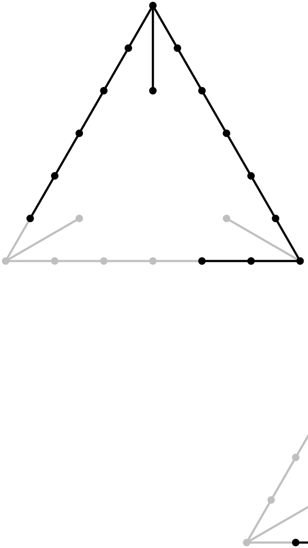
There is some hope though: we have reason to believe that the *irreducible components* of X all arise as birational modifications of semistable pairs, which gives us a way to study them.

Components of stable pairs

For instance, let $(\mathbb{P}^2, B)$ be a semistable pair (as classified by our earlier theorem). After a sequence of explicit birational modifications, we can obtain a surface (X', B') that is an irreducible component of a stable pair.

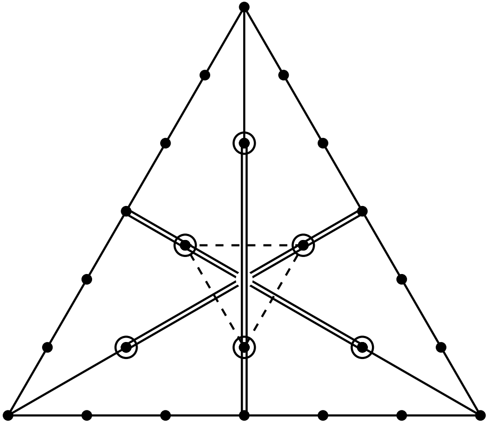
We can classify components of stable pairs that arise this way using the *complementary subdiagram* of their corresponding parabolic/hyperbolic subdiagram in Γ_{21}. The complementary subdiagram of a hyperbolic subdiagram is computed as follows (complementary subdiagrams for parabolic subdiagrams are also defined, but not so cleanly).

The dual diagram



The Vinberg-Scattone diagram

The resulting diagrams may be seen as subdiagrams of the *Vinberg-Scattone diagram*, Γ^{vs}, via the obvious inclusion $\Gamma_{21} \subset \Gamma^{\mathrm{vs}}$.



The six circled vertices here are called *inner vertices*. The three circled vertices in the corners are called *corner vertices*.

Classification of such components

Via this construction, components of stable pairs arising from semistable pairs $(\mathbb{P}^2, B)$ correspond, up to equisingular deformation, to connected parabolic/elliptic subdiagrams of the Vinberg-Scattone diagram that

- are not supported on the six inner vertices, and
- contain a corner vertex if and only if they contain the corresponding uncircled vertex connected to it.

The case $\mathbb{F}_4^0$

Moreover, it can be seen that this construction can also be applied to semistable pairs $(\mathbb{F}_4^0, B)$.

In this case, components of stable pairs arising from semistable pairs $(\mathbb{F}_4^0, B)$ correspond, up to equisingular deformations, to connected parabolic/elliptic subdiagrams of the Vinberg-Scattone diagram that

- are subdiagrams of the $\tilde{D}_{16}$ diagram, with its unique embedding into Γ^{vs}, and
- contain one of the two corner vertices contained in $\tilde{D}_{16}$ if and only if they contain the corresponding uncircled vertex connected to it.

Properties of this correspondence

This correspondence between components of stable pairs and parabolic/elliptic subdiagrams of the Vinberg-Scattone diagram has the following nice properties:

- The number of moduli of the component is equal to the number of vertices in the corresponding subdiagram.
- Degeneration corresponds to inclusion, with smaller subdiagrams being more degenerate.

Moreover, the parabolic/elliptic distinction here corresponds precisely to the Type II/Type III distinction for the corresponding K3 surfaces.

A conjecture

This leads to a natural conjecture:

Conjecture

There is a correspondence between irreducible components of stable pairs and connected parabolic/elliptic subdiagrams of Γ^{vs} *that are not supported on the six inner vertices.*

Compactifications

Now we come to an observation: The Vinberg-Scattone diagram is known to define a toroidal compactification of the moduli space of K3 surfaces of degree two.

Moreover, boundary components of this compactification are in bijective correspondence with parabolic/elliptic subdiagrams of Γ^{vs}, satisfying:

- The dimension of the boundary component is equal to the number of vertices in the corresponding subdiagram.
- Degeneration corresponds to inclusion, with smaller subdiagrams being more degenerate.
- The parabolic/elliptic distinction corresponds to the Type II/Type III distinction for degenerate K3 surfaces.

Another conjecture

This leads to an obvious conjecture:

Conjecture

The Hacking moduli space of stable pairs is closely related to the Vinberg-Scattone compactification of the moduli space of K3 surfaces of degree two.