# Degenerations of K3 Surfaces of Degree Two

Alan Thompson

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This talk is based upon my recent work on the explicit study of degenerations of K3 surfaces of degree two. Its contents may be found in more detail in the preprint [Tho10] and in my doctoral thesis [Tho11b], a copy of which is currently available on my website:

http://www.ualberta.ca/∼amthomps/

**Aim.** Study the geometric behaviour at the boundary of the moduli space $\mathcal{P}_2$ of polarised K3 surfaces of degree two (i.e. pairs $(S, D)$, where $S$ is a K3 surface and $D$ is a nef divisor on $S$ with $D^2 = 2$).

In order to do this, we begin by defining:

**Definition 1.** *A semistable degeneration of K3 surfaces $\pi\colon X \to \Delta$ (i.e. a proper, flat, surjective morphism $\pi\colon X \to \Delta$ whose general fibre $X_t = \pi^{-1}(t)$ for $t \in \Delta^* = \Delta - \{0\}$ is a smooth K3 surface, such that $X$ is smooth and $X_0 := \pi^{-1}(0)$ is reduced with normal crossings) equipped with a divisor $H$ on $X$ is called a* degeneration of K3 surfaces of degree two *if $H$ is effective and flat over $\Delta$, and $H$ induces a nef and big divisor $H_t$ on $X_t$ satisfying $H_t^2 = 2$ for all $t \in \Delta^*$. The divisor $H$ is called the* polarisation divisor *on $X$*

Let $\pi\colon X \to \Delta$ be a semistable degeneration of K3 surfaces of degree two with polarisation divisor $H$. Then Kulikov [Kul77] [Kul81] and Persson-Pinkham [PP81] show that we can perform birational modifications that affect only the central fibre $X_0$

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & \Delta & \end{tikzcd}$$

so that $\pi'\colon X' \to \Delta$ is semistable and has $\omega_{X'} \sim \mathcal{O}_{X'}$. Such $\pi'\colon X' \to \Delta$ is called a *Kulikov model* of our degeneration.

Kulikov models are classified by the following theorem:

**Theorem 2.** [Per77], [Kul77] [FM83] *Let $\pi\colon X \to \Delta$ be a semistable degeneration of K3 surfaces with $\omega_X \cong \mathcal{O}_X$, such that all components of $X_0$ are Kähler. Then either*

(I) *$X_0$ is a smooth K3 surface;*

(II) *$X_0$ is a chain of elliptic ruled components with rational surfaces at each end, and all double curves are smooth elliptic curves;*

(III) *$X_0$ consists of rational surfaces meeting along rational curves which form cycles in each component. If $\Gamma$ is the dual graph of $X_0$, then $|\Gamma|$, the topological support of $\Gamma$, is homeomorphic to the sphere $S^2$.*

This classification will form the first step in our study of the boundary of $\mathcal{P}_2$. We also have:

**Theorem 3.** [SB83, Theorem 1] *After a sequence of elementary modifications have been performed on $X_0$, we may further assume that $H$ is nef.*

Thus, from now on we may assume that we are in the following situation:

**Assumption 4.** *$\pi\colon X \to \Delta$ is a degeneration of K3 surfaces of degree two with polarisation divisor $H$, such that $\omega_X \cong \mathcal{O}_X$ and $H$ is nef.*

Using this, we have a naive description of the fibres at the boundary of $\mathcal{P}_2$:

- $X_0$ is a degenerate fibre of Type I, II or III;
- $H_0 = H \cap X_0$ is a nef divisor on $X_0$ with $H_0^2 = 2$.

We henceforth call these conditions $(*)$.

However, there is a problem with this description of the fibres on the boundary: Kulikov models of a given degeneration are not unique (i.e. the same $\pi^*\colon X^* \to \Delta^*$ can be completed to several different Kulikov models $\pi\colon X \to \Delta$). Elementary modifications can be used to move between these birationally equivalent models. This means that if we use the above description of the boundary to compactify our moduli space the resulting space will not be separated.

**Solution.** We proceed to the relative log canonical model of the pair $(X, H)$:

$$\phi\colon X - \to X^c := \mathbf{Proj}_\Delta \bigoplus_{n\geq 0} \pi_*\mathcal{O}_X(nH).$$

Results of the minimal model program show that $\phi$ defines a birational morphism over $\Delta^*$ and that all of the birationally equivalent Kulikov models map to the same relative log canonical model. So a better description of the fibres on the boundary of $\mathcal{P}_2$ would be "those pairs $((X^c)_0, (H^c)_0)$ that are the central fibres in the relative log canonical models of degenerations of K3 surfaces of degree two that satisfy the conclusion of Assumption 4".

It "just" remains to calculate these images.

**Lemma 5.** [Tho10, Lemma 4.1] *The map $\phi$ is a birational morphism and furthermore, writing*

$$(X_0)^c := \operatorname{Proj} \bigoplus_{n \geq 0} H^0(X_0, \mathcal{O}_{X_0}(nH_0))$$

*for the log canonical model of $X_0$, we have that $(X_0)^c$ and $(X^c)_0$ agree.*

*Sketch proof.* This is a consequence of the base point free theorem [Anc87] and the theorem on cohomology and base change. $\square$

In light of this, we set $X_0^c := (X_0)^c = (X^c)_0$. This allows us to restrict our attention to finding the log canonical models of pairs satisfying $(*)$.

**Example 6.** We begin by calculating the log canonical model when $X_0$ is a fibre of Type I (i.e. a smooth K3). Suppose first that $H_0$ is base point free. Then a simple Riemann-Roch calculation shows that $\phi_0 := \phi|_{X_0}$ is a birational morphism

$$\phi_0 \colon X_0 \longrightarrow X_0^c \cong X_6 \subset \mathbb{P}(1,1,1,3)$$

that contracts finitely many curves to Du Val singularities. This surface is the traditional "double cover of $\mathbb{P}^2$" that one normally associates with K3 surfaces of degree two. In analogy with curves of genus two, such K3 surfaces are called *hyperelliptic*.

**Example 7.** Suppose next that $H_0$ has base points. Then Mayer [May72] shows that $|2H_0|$ is base point free and a Riemann-Roch calculation shows that $\phi_0$ is a birational morphism

$$\phi_0 \colon X_0 \longrightarrow X_0^c \cong X_{2,6} \subset \mathbb{P}(1,1,1,2,3),$$

where the degree two relation does not involve the degree two variable, that contracts finitely many curves to Du Val singularities. Note that $X_0^c$ cannot be expressed as a double cover of $\mathbb{P}^2$. Instead, it can be seen as a double cover of the singular rational surface $X_2 \subset \mathbb{P}(1,1,1,2)$, which is isomorphic to the Hirzebruch surface $\mathbb{F}_4$ with the unique $(-4)$-curve contracted. Such K3 surfaces are called *unigonal*.

In fact, we find that these two cases are essentially all that can occur:

Table 1: $\phi(X_0) = \{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}(1,1,1,3)$ hyperelliptic.

| Type | Name | $f_6(x_i)$ | Comments |
|---|---|---|---|
| I | h | Reduced | $f_6$ has at worst A-D-E's. |
| II | 0h | Reduced | $f_6$ has one $\tilde{E}_7$, one $\tilde{E}_8$ or two $\tilde{E}_8$'s. |
| | 1 | $l^2(x_i)f_4(x_i)$ | $l$ linear, $\lvert l \cap f_4\rvert = 4$, $f_4$ may have an $\tilde{E}_7$. |
| | 2 | $q^2(x_i)f_2(x_i)$ | $q$ smooth quadric, $\lvert q \cap f_2\rvert = 4$. |
| | 3 | $f_3^2(x_i)$ | $f_3$ smooth cubic. |
| III | 0h | Reduced | $f_6$ has exactly one $T_{2,3,r}$ with $r \geq 7$ or $T_{2,q,r}$ with $q \geq 4$ and $r \geq 5$. |
| | 1 | $l^2(x_i)f_4(x_i)$ | $l$ linear, $\lvert l \cap f_4\rvert \leq 3$ with multiplicities $\leq 2$. |
| | 2 | $q^2(x_i)f_2(x_i)$ | $q$ (possibly nodal) quadric, $\lvert q \cap f_2\rvert \leq 4$ ($< 4$ if $q$ smooth) with multiplicities $\leq 2$. |
| | 3 | $f_3^2(x_i)$ | $f_3$ cubic with nodal singularities. |

**Theorem 8.** [Tho10, Theorem 3.1] *Let $\pi\colon X \to \Delta$ be a semistable degeneration of K3 surfaces, with $\omega_X \cong \mathcal{O}_X$. Let $H$ be a divisor on $X$ that is effective, nef and flat over $\Delta$, and suppose that $H$ induces a nef and big divisor $H_t$ on $X_t$ satisfying $H_t^2 = 2$ for $t \in \Delta^*$.*

*Then the morphism $\phi\colon X \to X^c$ taking $X$ to the relative log canonical model of the pair $(X, H)$ maps $X_0$ to one of:*

- (*Hyperelliptic Case*) *A sextic hypersurface*

$$\{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x, x_3, z].$$

- (*Unigonal Case*) *A complete intersection*

$$\{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$

  *where $f_6(0,0,0,1) \neq 0$.*

Furthermore, the possible central fibres are classified in tables 1 and 2.

Note the relationship between the entries in these tables and other known compactifications of the moduli space of K3 surfaces of degree two (fully explained in [Tho10, Subsection 3.1]):

- (II.1)-(II.4) correspond to the four Type II boundary components appearing the Baily-Borel-Satake compactification [Fri84]. The case (II.0) maps to two different components, depending upon whether an $\tilde{E}_7$ or $\tilde{E}_8$ singularity is present.

Table 2: $\phi(X_0) = \{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}(1,1,1,2,3)$ unigonal.

| Type | Name | $f_2(x_i)$ | Comments |
|---|---|---|---|
| I | u | Irreducible | $\phi(X_0)$ has at worst RDP's. |
| II | 0u | Irreducible | $\phi(X_0)$ has one $\tilde{E}_7$, one $\tilde{E}_8$ or two $\tilde{E}_8$'s. |
| | 4 | $l_1(x_i)l_2(x_i)$ | $l_i$ linear, $\lvert l_1 \cap l_2 \cap f_6\rvert = 3$, where $\phi(X_0)$ may have one or two $\tilde{E}_8$'s. |
| III | 0u | Irreducible | $\phi(X_0)$ has exactly one $T_{2,3,r}$ with $r \geq 7$ or $T_{2,q,r}$ with $q \geq 4$ and $r \geq 5$. |
| | 4 | $l_1(x_i)l_2(x_i)$ | $l_i$ linear, $\lvert l_1 \cap l_2 \cap f_6\rvert = 2$, where the curve $\{f_6 = l_i = 0\}$ may be non-reduced for one or both choices of $i \in \{1, 2\}$. |

- All cases appear in Shah's [Sha80] GIT compactification, although several of our cases map to the same GIT points.

*Sketch proof of Theorem 8.* Recall that, by Lemma 5, we just have to analyse the log canonical model of the pair $(X_0, H_0)$. Write $X_0$ as a union of irreducible components $X_0 = V_1 \cup \cdots \cup V_r$ and let $H_i = H \cap V_i$. Then we have:

**Lemma 9.** [Tho10, Lemma 7.1] *If $H_i^2 = 0$, then $V_i$ is contracted by $\phi$.*

This allows us to focus our attention on components $V_i$ with $H_i^2 > 0$. We have:

**Theorem 10.** [Tho10, Theorem 7.2] *After performing a birational modification on $X_0$ that does not affect the form of its log canonical model, we may assume that for any surface $V_i$ with $H_i^2 > 0$, the linear system $|nH_i|$*

- *has no fixed components or base locus for $n \geq 2$;*
- *defines a morphism to projective space that is birational onto its image for $n \geq 3$.*

*Sketch proof.* This follows from known facts about anticanonical pairs [Fri83] and elliptic ruled surfaces [Tho10, Subsection 4.1] if one can prove that $\mathrm{Fix}(|H_i|)$ does not contain any component of the double locus on $V_i$. This can be proved for the central fibre of a degeneration of K3 surfaces of degree two, but the proof does not work for other polarisations (it relies upon the fact that the only partitions of 2 are (2) and $(1, 1)$). □

To finish proving the theorem, one just has to explicitly calculate cases corresponding to different positions of surfaces that have $H_i^2 > 0$ within $X_0$. For instance, in the Type II case we have 5 possibilities:

1. There is one component $V_i$ with $H_i^2 = 2$, that is rational. This case gives rise to cases (II.0) (where there is exactly one $\tilde{E}_7$ or $\tilde{E}_8$ singularity present), (II.1) and (II.2), distinguished by the intersection number $H_i.K_{V_i}$.

2. There is one component $V_i$ with $H_i^2 = 2$, that is elliptic ruled. This case gives rise to cases (II.0) (where there are two $\tilde{E}_8$ singularities present) and (II.1) (where $\phi(X_0)$ has an $\tilde{E}_7$ singularity).

3. There are two components $V_i$ and $V_j$ with $H_i^2 = H_j^2 = 1$, that are both rational. This gives rise to cases (II.3) and (II.4).

4. There are two components $V_i$ and $V_j$ with $H_i^2 = H_j^2 = 1$, one of which is rational and the other of which is elliptic ruled. This case gives rise to case (II.4) (where $\phi(X_0)$ has exactly one $\tilde{E}_8$ singularity).

5. There are two components $V_i$ and $V_j$ with $H_i^2 = H_j^2 = 1$, that are both elliptic ruled. This case gives rise to case (II.4) (where there are two $\tilde{E}_8$'s present).

$\square$

As well as the study of the moduli of K3 surfaces of degree two, this result has applications to the construction of threefolds fibred by K3 surfaces of degree two, by giving explicit constraints on the types of singular fibres that can arise. This is explored in the preprint [Tho11a].

More specifically, let $X$ be a (possibly slightly singular) threefold that admits a semistable fibration $\pi : X \to S$ over a nonsingular curve $S$, with general fibre a smooth K3 surface. Let $\mathcal{L}$ be a line bundle on $X$ that induces a nef and big line bundle with self-intersection number two on a general fibre.

Given this, we see that any such K3-fibred threefold uniquely determines a certain 5-tuple of data on the base curve $S$, and that from this data the relative log canonical model of the polarised variety $(X, \mathcal{L})$ over $S$ can be explicitly reconstructed. Briefly, this construction works by defining a rational surface bundle on $S$ with fibres $\mathbb{P}^2$ and $X_2 \subset \mathbb{P}(1,1,1,3)$, then taking a double cover of this to obtain the relative log canonical model of $X$.

Furthermore we see that, under certain conditions, any such 5-tuple of data arises from some threefold fibred by K3 surfaces of degree two and can be used to reconstruct its relative log canonical model. This is a K3 surface analogue of Catanese and Pignatelli's main result [CP06, Theorem 4.13]. In particular, it proves a complete characterisation of the threefolds that can occur as relative log canonical models of threefolds fibred by K3 surfaces of degree two, and gives a method to construct them explicitly.

Finally, a less obvious application is to the study of threefolds fibred by K3 surfaces that admit a polarisation by the hyperbolic plane lattice $H$. It is not difficult to prove that the general $H$-polarised K3 surface is exactly the unigonal K3 from Example 7, and that any unigonal K3 admits an $H$-polarisation. So by restricting our main result to the case where the general fibre is unigonal, we may obtain a classification of degenerate $H$-polarised K3 surfaces. A modification of the methods described above should then allow the explicit construction of threefolds fibred by K3 surfaces admitting an $H$-polarisation.

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