Degenerations Of Surfaces With Kodaira Number Zero

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References

This talk is based upon the article *The birational geometry of degenerations: an overview* by R. Friedman and D. Morrison, contained in the book *The birational geometry of degenerations*, edited by R. Friedman and D. Morrison. Full references for all results quoted in this talk may be found there.

1 What is a degeneration?

Definition. *A degeneration of compact complex manifolds of dimension n is a proper map $\pi : X \to \Delta$, where X is a complex manifold and $\Delta = \{z \in \mathbb{C} \mid 0 \leq |z| < 1\}$ is the unit disc, such that the restriction $\pi' : X^* \to \Delta^*$ (where $\Delta^* := \Delta \backslash \{0\}$) is a smooth map with fibres of dimension n.*

Remark In this talk we frequently switch back and forth between discussing compact complex manifolds and algebraic varieties. We will try to make it clear which category we are working in when it is important to do so.

2 Why do we study them?

- To compactify moduli spaces of smooth varieties.
- To study singularities: If a singularity occurs in the central fibre of a degeneration, we can learn a lot about the singularity by studying the degeneration. For example, we know that the singularity is smoothable.
- To assist in the birational classification of smooth varieties: By Iitaka's fibration, one is lead to consider varieties V that admit a morphism

$f : V \to W$ whose general fibre is a smooth connected variety with Kodaira dimension $\kappa = 0$. The properties of V (the canonical class, for example) are closely linked to the properties of the singular fibres in such a fibration, which can be studied using the theory of degenerations.

This talk will primarily be concerned with the third case.

3 What do we want to know?

One of the fundamental problems in degeneration theory is that of finding a "good" birational model of a degeneration to study. We will investigate this further in the case where the general fibre has $\kappa = 0$.

So let $\pi : X \to \Delta$ be a degeneration of algebraic varieties of dimension n (where $n = 1$ or 2). By Horikawa's theorem, we may assume that X is smooth and $\pi^{-1}(0) = X_0$ is a divisor with normal crossings.

These are still not easy to study, because the central fibre is not necessarily reduced, which makes the problem very combinatorially difficult (we will discuss this later; compare Kodaira's classification of singular elliptic curves).

However, many of the useful properties (e.g. properness) of the degeneration are preserved under *base change of order* m, defined by the diagram:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \Delta & \xrightarrow{\sigma} & \Delta \end{array}$$

where the map σ is given by:

$$\sigma : t \longmapsto t^m.$$

Note here that X' may be singular, even if X is not.

Furthermore, we have:

Theorem (Knudsen-Mumford-Waterman '73)**.** *Given $\pi : X \to \Delta$, there exists m such that, if $\pi' : X' \to \Delta$ is the base change of order m, there is a birational modification Y of X' such that Y is smooth and Y_0 is reduced with only normal crossings.*

We have a diagram:

$$\begin{array}{ccccc} Y & \longrightarrow & X' & \longrightarrow & X \\ {\scriptstyle \rho}\big\downarrow & & {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \Delta & = & \Delta & \longrightarrow & \Delta \end{array}$$

We call $\rho : Y \to \Delta$ a *semistable* degeneration (note that this is related to semistability in the sense of compactifying moduli spaces).

In light of this, we will restrict our attention to the study of semistable degenerations.

4 Elliptic curves

Let $\pi : X \to \Delta$ be a degeneration of curves such that $\pi^{-1}(t) = X_t$ is a smooth curve of genus 1 (not necessarily semistable).

The first obvious “good” model to take for the degeneration is the relative minimal model, obtained by contracting all (-1)-curves contained in a fibre. The singular fibres of such a model were famously classified by Kodaira.

In the case where the fibration has a section, there is also a second candidate for a “good” model: The Weierstass model, that is obtained by contracting all components of X_0 that do not meet the section. In this case, the total space may have rational double point singularities. However, Weierstrass models have a nice explicit construction (due to Nakayama) that makes them easy to study.

Finally, we note what happens when we proceed to the semistable degeneration. In this case, Kodaira's classification simplifies and there are only two possibilities for the central fibre of the minimal model: A smooth elliptic curve or a cycle of rational curves of self-intersection (-2). This highlights nicely the combinatorial advantage of the semistable assumption.

5 The surface case

Let $\pi : X \to \Delta$ be a semistable degeneration of surfaces such that $\pi^{-1}(t) = X_t$ is a smooth surface with $\kappa = 0$.

The first “good” model that we may consider is again the minimal model. However, there is a problem with this: The total space may be singular, with terminal singularities. These have been listed and classified, but the classification is rather complex (See *Young person's guide to canonical singularities*, by M. Reid). For this reason, we seek a nonsingular model (in fact, this new model will turn out to be related to the minimal model in certain cases).

We have:

Theorem (Kulikov '77, '81, Persson-Pinkham '81). *If $\pi : X \to \Delta$ is a semistable degeneration of surfaces with $K_{X_t} \sim 0$ for $t \neq 0$ (i.e. X_t is a K3 surface, complex torus or Kodaira surface) and if all components of X_0 are*

Kähler, then there exists a birational modification X' of X, isomorphic to X over Δ^*, such that $K_X \sim 0$.

This suggests that, for semistable degenerations of surfaces with $\kappa = 0$, we may be able to make the pluricanonical bundle of the total space trivial, giving the "good" model we are looking for. Unfortunately, there exist degenerations $\pi : X \to \Delta$ with $mK_{X_t} \sim 0$ for some $m > 0$, such that all components of X_0 are Kähler but mK_Y is not trivial for any birational modification Y of X. To solve this problem, we shall restrict our attention to those cases where the pluricanonical bundle can be made trivial, and call a model X a *Kulikov model* if $mK_X \sim 0$.

It is possible to completely classify the central fibres of Kulikov models, but first we need a technical definition:

Definition. *Let $\pi : X \to \Delta$ be a semistable degeneration of surfaces. Write $X_0 = \bigcup_{i=1}^N V_i$, for irreducible components V_i. Define the* dual graph Γ *of X_0 as follows: Γ is a simplicial complex whose vertices $P_1, \ldots, P_N$ correspond to the components $V_1, \ldots, V_N$ of X_0. The simplex $\langle P_{i_1}, \ldots, P_{i_m} \rangle$ belongs to Γ if and only if $V_{i_i} \cap \cdots \cap V_{i_m} \neq \emptyset$.*

Classification Theorem (Kulikov '77, Persson '77)**.** *Let $\pi : X \to \Delta$ be semistable with $mK_X \sim 0$ for some $m > 0$, such that all components of X_0 are Kähler. Then either:*

Type I *X_0 is smooth.*

Type II *X_0 is a cycle of elliptic ruled components or a chain of elliptic ruled components (possibly with rational surfaces at one or both ends of the chain) and all double curves are smooth elliptic.*

Type III *X_0 consists of rational surfaces meeting along rational curves that form cycles in each component. The dual graph Γ of X_0 has support $|\Gamma|$ equal to S^2, $\mathbb{RP}^2$, $S^1 \times S^1$ or K^2 (the Klein bottle).*

We have the following table:

Type	$\lvert\Gamma\rvert$	X_t
I	Point	Same as X_0
II	Interval, both ends elliptic	Hyperelliptic
	Interval, one end rational	Enriques
	Interval, both ends rational	K3
	S^1	Complex torus, hyperelliptic or Kodaira (primary or secondary)
III	S^2	K3
	$\mathbb{RP}^2$	Enriques
	$S^1 \times S^1$	Complex torus or 1° Kodaira
	K^2	Hyperelliptic or 2° Kodaira

Unfortunately, Kulikov models are not unique. We illustrate this with an example:

Example Consider the degeneration:

$$X := \{z^2 = g_3(w,x,y)^2 + tf_6(w,x,y)\} \subset \mathbb{P}(1,1,1,3) \times \Delta$$

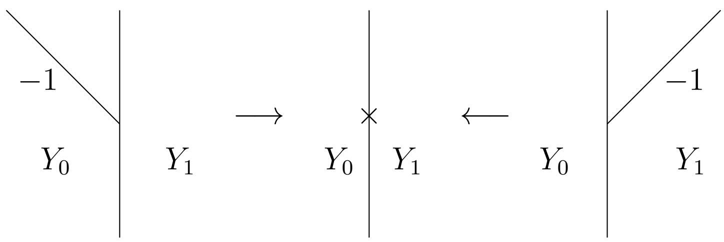
The general fibre X_t is a sextic in $\mathbb{P}_{(1,1,1,3)}[w,x,y,z]$, i.e. a $\langle 2 \rangle$-polarised K3 surface. X_0 is the locus $\{z^2 = g_3(w,x,y)^2\}$, which is the union of the two (nonsingular) surfaces:

$$\begin{aligned} Y_0 &:= \{z = g_3(w,x,y)\} \\ Y_1 &:= \{z = -g_3(w,x,y)\} \end{aligned}$$

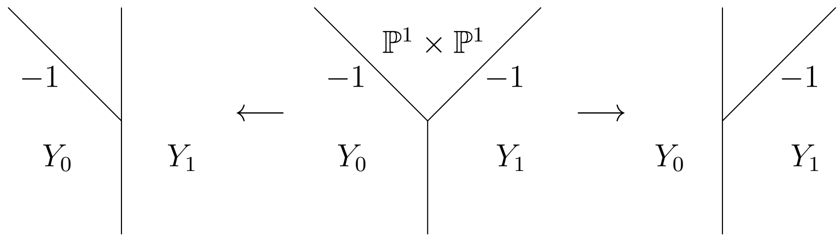
These are rational surfaces, meeting along the elliptic curve $g_3(w,x,y) = 0$. However, X is singular, with 18 singular points lying on the intersection $Y_0 \cap Y_1 \subset X_0$.

A Kulikov model for X is obtained by resoving these singularities. We may do this by blowing them up in the fibre X_0. But we have a choice to make: Do we blow them up in Y_0 or Y_1?

This leads to the equivalent models:



We may pass between them with an elementary modification of Type I:



Where the first morphism blows up the (-1)-curve in Y_0 and the second morphism contracts $\mathbb{P}^1 \times \mathbb{P}^1$ along its other ruling.

There are also elementary modifications of Types 0 and II, but we have no time to study them here. They are important because of the following result:

Theorem (Shepherd-Barron '83)**.** *If $X \to \Delta$ and $X' \to \Delta$ are two birationally equivalent Kulikov models of a degeneration of polarised K3 surfaces, then X' is obtained from X by a sequence of elementary modifications of Types 0, I and II.*

6 A few loose ends

We mentioned earlier that in certain cases the Kulikov model is related to the minimal model. In fact, for Type II degenerations, Crauder (*Minimal models and degenerations of surfaces with Kodaira number zero*, 1994) has proved that the minimal model may be obtained from the Kulikov model by the careful contraction of components in X_0. Unfortunately, contracting the wrong components can lead to non-terminal singularities.

Finally, we would also like to mention briefly the status of the classification in the non-reduced (i.e. non-semistable) case. This classification was completed in the Type II case by Crauder (*Triple-point-free degenerations of surfaces with Kodaira number zero*, in the book *The birational geometry of degenerations*). However, his classification is extremely combinatorially complex, which suggests that a similar result for Type III degenerations is somewhat unfeasible.