The Construction Of Ample $\langle 2 \rangle$-Polarised K3-Fibrations

Alan Thompson

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This talk is based upon the preprint *A Model For Ample $\langle 2 \rangle$-Polarised K3-Fibrations*, which is available from the preprints section of my website:

http://people.maths.ox.ac.uk/∼thompsona

Full proofs of all results (or references to them) may be found there.

This talk aims to define *ample $\langle 2 \rangle$-polarised K3-fibrations*, then provide a general method to construct them. We begin with some information about the Picard group of a K3 surface, which will enable to define the notion of a polarisation. First, however, we recall the definition of a K3 surface:

Definition 1 *A* K3 surface *is a nonsingular, projective, algebraic surface X with $K_X \sim 0$ and $h^1(X, \mathcal{O}_X) = 0$.*

Consider the cohomology group $H^2(X, \mathbb{Z})$. This group is torsion-free and can be given the structure of a lattice, with bilinear form induced by the cup-product pairing. This lattice has signature (3,19) and is isometric to the *K3 lattice*

$$L := (-E_8) \oplus (-E_8) \oplus H \oplus H \oplus H,$$

where H denotes the hyperbolic plane (an even, unimodular, indefinite lattice of rank 2) and E_8 is the root lattice E_8 (an even, unimodular positive definite lattice of rank 8).

Now, by the exponential cohomology sequence

$$\cdots \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots$$

we obtain an injective map $\mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z})$. This realises $\mathrm{Pic}(X)$ as a sublattice of L of signature $(1, \rho(X) - 1)$ (where $\rho(X)$ denotes the rank of the Picard group of X).

The ample divisor classes in $\mathrm{Pic}(X)$ are precisely those that map to the Kähler classes in $H^2(X,\mathbb{R})$ (which contains $H^2(X,\mathbb{Z})$ as a subgroup). Let $\mathcal{C}_X^+$ denote the Kähler cone in $H^2(X,\mathbb{R})$. Then the ample divisor classes are given by

$$\mathrm{Pic}(X)^+ := \mathcal{C}_X^+ \cap H^2(X,\mathbb{Z}).$$

Now we are in a position to start talking about polarisations of K3 surfaces. Let M be an even, non-degenerate lattice of signature (1,t) (for $1 \leq t \leq 19$). Then we have:

Definition 2 *An* M-polarised K3 surface *is a pair* (X,j)*, where* X *is a K3 surface and* $j : M \hookrightarrow \mathrm{Pic}(X)$ *is a primitive lattice embedding.*

We would now like to have a criterion for $j(M)$ to contain an ample divisor class. If this happens, we can say that (X,j) is *ample* M*-polarised*. To do this, we emulate the construction of the cone of Kähler classes in $H^2(X,\mathbb{Z})$ to get a subset $\mathcal{C}_M^+ \subset M$ (however, this construction is not terribly enlightening so we omit the details from this talk). Then we say:

Definition 3 (X,j) *is called* ample M-polarised *if*

$$j(\mathcal{C}_M^+) \cap \mathrm{Pic}(X)^+ \neq \emptyset.$$

Example 4 Let M be the lattice $\langle 2 \rangle$. This is a free $\mathbb{Z}$-module of rank 1, generated over $\mathbb{Z}$ by e with $\langle e,e \rangle = 2$. Let (X,j) be an ample $\langle 2 \rangle$-polarised K3 surface. Then $j(e)$ corresponds to a linear system of ample divisors $|D|$ on X. Assuming $|D|$ to be base point free (the generic case), a simple application of Riemann-Roch gives that X can be realised as a sextic hypersurface in the weighted projective space $\mathbb{P}(1,1,1,3)$.

We next want to construct a fibration by such surfaces. An obvious way to do this is to begin with a fibration by weighted projective spaces $\mathbb{P}(1,1,1,3)$, then take a sextic hypersurface in each in a way that gives a flat family over the base. To get this weighted projective fibration, we define:

Definition 5 *Let* S *be a scheme and let* $(a_0,\dots,a_n)$ *be a sequence of strictly positive integers. Define a weighted locally free sheaf with weights* $(a_1,\dots,a_n)$ *to be a locally free sheaf of* $\mathcal{O}_S$*-modules* $\mathcal{E}$ *together with an ordered decomposition* $\mathcal{E} \cong \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_n$ *such that each* $\mathcal{E}_i$ *is a locally free sheaf and such that the direct sum is to be interpreted as a graded sheaf with* $\mathcal{E}_i$ *placed in degree* a_i *for* $0 \leq i \leq n$*.*

Definition 6 *Let S be a scheme. Given a weighted locally free sheaf $\mathcal{E}$ with weights $(a_0, \ldots, a_n)$, let $\widetilde{\mathrm{Sym}}(\mathcal{E})$ denote the weighted symmetric algebra of $\mathcal{E}$, where we insist that $\mathcal{E}_i$ have homogeneous degree a_i in $\widetilde{\mathrm{Sym}}(\mathcal{E})$. We define the weighted projective bundle associated to $\mathcal{E}$ to be the S-scheme*

$$\widetilde{\mathbb{P}}_S(\mathcal{E}) := \mathbf{Proj}(\widetilde{\mathrm{Sym}}(\mathcal{E})) \xrightarrow{p} S$$

The first thing that we want to know about weighted projective bundles is that the fibres are the weighted projective spaces that we expect. We have:

Lemma 7 (Mullet '06) *Let S be a nonsingular variety over $\mathbb{C}$ and let $\mathcal{E} \cong \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_n$ be a weighted locally free sheaf with weights $(a_0, \ldots, a_n)$. Then the weighted projective bundle $\widetilde{\mathbb{P}}_S(\mathcal{E})$ is a locally trivial fibre bundle over S with fibre the weighted projective space $\mathbb{P}(a_0, \ldots, a_0, a_1, \ldots, a_{n-1}, a_n, \ldots, a_n)$, where each a_i appears with multiplicity* $\mathrm{rank}(\mathcal{E}_i)$.

Now we are finally ready to begin the construction of our $\langle 2 \rangle$-polarised K3-fibrations. First, however, we need to define them. We begin by letting M be an even, non-degenerate lattice of signature $(1, t)$, for $0 \leq t \leq 19$. Then we define:

Definition 8 *Let S be a normal complex variety. An* ample M-polarised K3-fibration *of S, denoted (X, π, j), consists of:*

1. *A normal complex variety X,*
2. *A projective, flat, surjective morphism $\pi : X \to S$ with connected fibres, whose general fibres are K3 surfaces,*
3. *A $\mathbb{Z}$-module homomorphism $j : M \to \mathrm{Pic}(X/S)$ that induces a primitive lattice embedding $j_F : M \to \mathrm{Pic}(F)$ on a general fibre F, making (F, j_F) into an ample M-polarised K3 surface.*

Now let S be a nonsingular curve and let M be the lattice $\langle 2 \rangle$. Let $\mathcal{E}$ be a rank 3 vector bundle and $\mathcal{L}$ a line bundle on S. Treat $\mathcal{E} \oplus \mathcal{L}$ as a weighted locally free sheaf with weights (1,3) and define $Y := \widetilde{\mathbb{P}}_S(\mathcal{E} \oplus \mathcal{L})$. Let $p : Y \to S$ be the natural projection and $\mathcal{O}_Y(1)$ be the tautological line bundle on Y. Then, by Lemma 7, Y is a locally trivial fibre bundle on S with fibre $\mathbb{P}(1, 1, 1, 3)$.

We next want to construct a divisor on Y which intersects the general fibre in a hypersurface of degree 6. So consider $\mathbb{P}_S(\mathrm{Sym}^6(\mathcal{E}) \oplus \mathcal{O}_S)$. There is an open embedding $\mathrm{Sym}^6(\mathcal{E}) \hookrightarrow \mathbb{P}_S(\mathrm{Sym}^6(\mathcal{E}) \oplus \mathcal{O}_S)$ given by $v \mapsto [v, 1]$. We henceforth identify $\mathrm{Sym}^6(\mathcal{E})$ with its image under this embedding. Let

$$a : \mathcal{L}^2 \longrightarrow \mathbb{P}_S(\mathrm{Sym}^6(\mathcal{E}) \oplus \mathcal{O}_S)$$

be a sheaf homomorphism such that $\mathrm{Im}(a) \cap \mathrm{Sym}^6(\mathcal{E}) \neq \emptyset$. Then there exists an open set $S_0 \subset S$ such that $\mathrm{Im}(a) \subset \mathrm{Sym}^6(\mathcal{E})$ on S_0. Hence, restricting to S_0, we have $a \in \Gamma(S_0, \mathcal{L}^{-2} \otimes \mathrm{Sym}^6(\mathcal{E}))$. Let $p_0 : Y_0 \to S_0$ denote the restriction of p to S_0. Let $a' \in \Gamma(Y_0, p_0^*\mathcal{L}^{-2} \otimes \mathrm{Sym}^6(p_0^*\mathcal{E}))$ denote the inverse image of a under p_0.

Now let Z and T be the sections of $p^*\mathcal{E}^\vee \otimes \mathcal{O}_Y(1)$ and $p^*\mathcal{L}^{-1} \otimes \mathcal{O}_Y(3)$ corresponding to the natural morphisms:

$$\begin{array}{rcl} p^*\mathcal{E} & \longrightarrow & \mathcal{O}_Y(1) \\ p^*\mathcal{L} & \longrightarrow & \mathcal{O}_Y(3). \end{array}$$

Denote by $W_0^3(\mathcal{E}, \mathcal{L}, a)$ the divisor on Y_0 defined by the equation $T^2 - a'Z^6$. Then $W_0 := W_0^3(\mathcal{E}, \mathcal{L}, a)$ is flat over S_0, so there exists a unique closed subscheme $W^3(\mathcal{E}, \mathcal{L}, a) \subset Y$ whose restriction to Y_0 is W_0.

Definition 9 *$W^3(\mathcal{E}, \mathcal{L}, a)$ is called the* 3rd family K3-Weierstrass model *over S of type $(\mathcal{E}, \mathcal{L}, a)$.*

$W := W^3(\mathcal{E}, \mathcal{L}, a)$ has the following properties:

1. W is a normal complex variety and $p : W \to S$ is a projective, flat, surjective morphism whose general fibres are irreducible sextic hypersurfaces in $\mathbb{P}(1, 1, 1, 3)$.

2. Let σ be a section of $p^*\mathcal{E}$. Let σ' be the push-forward of σ by the morphism $p^*\mathcal{E} \to \mathcal{O}_Y(1)$. Define $\Sigma^3(\mathcal{E}, \mathcal{L}, a) := W \cap (\sigma')_0$. Then, on a general fibre W_s, the divisor $\Sigma_s = \Sigma^3(\mathcal{E}, \mathcal{L}, a) \cap W_s$ defines a 2:1 morphism $W_s \to \mathbb{P}^2$, branched over a sextic curve, under which Σ_s is the inverse image of a hyperplane section. So Σ_s is an ample divisor on W_s with self-intersection number 2.

Finally, we have:

Theorem 10 *Let S be a nonsingular curve. Let e be a generator of $\langle 2 \rangle$ as a free $\mathbb{Z}$-module. Let (X, π, j) be an ample $\langle 2 \rangle$-polarised K3-fibration of S and let D be the divisor on X defined by $D := j(e)$. Suppose that, on a general fibre F, $|D|_F|$ is a base point free linear system. Then there exists a 3rd family K3-Weierstrass model $W^3(\mathcal{E}, \mathcal{L}, a)$ over S and a birational map $\mu : X - \to W^3(\mathcal{E}, \mathcal{L}, a)$ over S such that $\mu_*(D) \sim \Sigma^3(\mathcal{E}, \mathcal{L}, a)$.*