Hodge Theory, Period Mappings And The Moduli Of K3's

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References
All results referenced in this talk may be found in the book *Compact Complex Surfaces* (2nd Edition), by Barth, Hulek, Peters and Van de Ven. All result numbers will refer to this book, unless otherwise stated.

1 Hodge Theory

Let X be a compact Kähler manifold. Define Ω_X^p to be the sheaf of germs of holomorphic p-forms on X. Then define

$$H^{p,q}(X) := H^q(\Omega_X^p)$$

(This is not the formal definition, but is equivalent in our case by Dolbeault's isomorphism, Section I.12). We let $h^{p,q}(X) := \dim H^{p,q}(X)$. Then we have:

Theorem I.13.4. (Hodge Decomposition)

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

This is useful because it relates topological and sheaf cohomology on X. It is an example of a Hodge structure.

2 K3 Surfaces

A *K3 surface* is a compact complex surface X with trivial canonical class $K_X = 0$ and zero first Betti number $b_1(X) = \dim H^1(X, \mathbb{C}) = 0$. The first thing we'd like to know about K3 surfaces is the form of their cohomology and, more specifically, its Hodge decomposition.

Let L denote the lattice

$$L := (-E_8) \oplus (-E_8) \oplus H \oplus H \oplus H.$$

It is a free $\mathbb{Z}$-module of rank 22, with bilinear form $(-,-)$ of signature $(3,19)$ given by the above decomposition.

Furthermore, let $L_{\mathbb{C}} := L \otimes \mathbb{C}$.

Proposition VIII.3.3. *Let X be a K3 surface. Then*

- $H^1(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$
- $H^2(X,\mathbb{Z})$ *is torsion free of rank 22 and, when equipped with the cup-product pairing, isometric to L*

From this proposition, we note that any K3 surface must be Kähler (any 2-form must be closed, since $H^3(X,\mathbb{C}) = 0$; the positivity condition is slightly harder to show, but can be proven by examining the lattice L). So we have a Hodge decomposition:

Proposition VIII.3.4. *Let X be a K3 surface. Then*

- $h^{1,0}(X) = h^{0,1}(X) = h^{2,1}(X) = h^{1,2}(X) = 0$
- $h^{2,0}(X) = h^{0,2}(X) = 1$
- $h^{1,1}(X) = 20$

3 The Period Map for K3's

For $\omega \in L_{\mathbb{C}}$, we denote by $[\omega] \in \mathbb{P}(L_{\mathbb{C}})$ the corresponding line (i.e. the line $\mathbb{C}.\omega$, considered as a point in $\mathbb{P}(L_{\mathbb{C}})$) and set

$$\Omega := \{[\omega] \in \mathbb{P}(L_{\mathbb{C}}) \mid (\omega,\omega) = 0,\ (\omega,\overline{\omega}) > 0\}.$$

Ω is called the *period domain.*

A *marked K3 surface* (X,ϕ) is a K3 surface X along with a choice of isometry $\phi : H^2(X,\mathbb{Z}) \to L$. Since $h^{2,0}(X) = 1$, the complexified map $\phi_{\mathbb{C}} : H^2(X,\mathbb{C}) \to L_{\mathbb{C}}$ takes the nowhere vanishing holomorphic 2-forms (i.e. elements of $H^2(X,\Omega^2_X)$) to a line in $L_{\mathbb{C}}$, which determines a point in $\mathbb{P}(L_{\mathbb{C}})$.

As such a 2-form ω_X has $(\omega_X,\omega_X) = 0$ and $(\omega_X,\overline{\omega}_X) > 0$ (by the definition of L), (X,ϕ) corresponds to a point in Ω.

Looking at this, Ω seems like a good candidate for a moduli space of marked K3's.

4 The Torelli Theorem and the Moduli of K3's

If X and X' are K3 surfaces, an isomorphism of $\mathbb{Z}$-modules

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X', \mathbb{Z})$$

is called a *Hodge isometry* if

1. It preserves the cup-product, and
2. Its $\mathbb{C}$-linear extension $H^2(X, \mathbb{C}) \to H^2(X', \mathbb{C})$ preserves the Hodge decomposition.

Then we have:

Theorem VIII.11.2. (Weak Torelli) *Two K3 surfaces X and X' are isomorphic if there exists a Hodge isometry $\phi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$.*

Corollary. *Two K3 surfaces are isomorphic if and only if there are markings for them such that the corresponding period points are the same.*

Theorem VIII.14.2. (Surjectivity of the Period Map) *Every point of Ω occurs as the period point of a marked K3 surface*

If one makes additional assumptions on the form of ϕ, then one can further show that the isomorphism in the weak Torelli theorem is unique. This gives the Torelli theorem for K3's.

The weak Torelli theorem gives us that Ω forms a coarse moduli space for marked K3 surfaces. But, using the full Torelli theorem, we may try to go further and construct a fine moduli space for marked K3's. Unfortunately, following the construction here, the candidate moduli space that we get is not separated. In order to solve this we must add additional structure, in the form of *marked pairs*. Unfortunately, there is no time to cover these in this talk; full details may be found in Section VIII.12.