Moduli of K3 Surfaces

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12th September 2013.
Concentrated Graduate Course preceeding Workshop 1 on
Modular Forms around String Theory.
Fields Institute, Toronto.

For the interested reader, all of the results mentioned in this talk may be found in either [BHPvdV04, Chapter VIII] or [Sca87].

1 Hodge theory for K3 surfaces

Throughout this talk, X will denote an arbitrary K3 surface.

Recall that the Hodge diamond of any K3 surface looks like

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array},$$

so that all the interesting behaviour happens in the second cohomology group. As we shall see, the structure of this cohomology group determines the isomorphism class of a K3 surface, so can be used to construct a moduli space for K3 surfaces.

The second cohomology group $H^2(X,\mathbb{Z})$ with the cup-product pairing forms a lattice isometric to the *K3 lattice*

$$\Lambda_{\mathrm{K3}} := H \oplus H \oplus H \oplus E_8 \oplus E_8,$$

where H is the hyperbolic plane (an even, unimodular, indefinite lattice of rank 2) and E_8 is the even, unimodular, negative definite lattice of rank 8 corresponding to the Dynkin diagram E_8. The lattice Λ_{K3} has rank 22 and signature $(3,19)$. We then have:

Theorem 1 (Weak Torelli)**.** *Two K3 surfaces X and X' are isomorphic if and only if there is a lattice isometry $H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$, whose $\mathbb{C}$-linear extension $H^2(X,\mathbb{C}) \to H^2(X',\mathbb{C})$ preserves the Hodge decomposition (such an isometry is called a* Hodge isometry*).*

2 The period mapping

We can use the weak Torelli theorem to begin defining a moduli space for K3 surfaces. We begin by defining a *marking* on X.

Definition 2. A *marking* on X is a choice of isometry $\phi\colon H^2(X,\mathbb{Z}) \to \Lambda_{\mathrm{K3}}$.

Let $\omega \in H^{2,0}(X) = H^0(X, \Omega_X^2)$ be any class. Then $(\omega, \omega) = 0$ and $(\omega, \bar{\omega}) > 0$. So, if ϕ is a marking for X and $\phi_{\mathbb{C}}$ is its complexification, then $\phi_{\mathbb{C}}(H^{2,0}(X))$ defines a point in

$$\Omega_{\mathrm{K3}} := \{[\omega] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}.$$

Ω_{K3} is a 20-dimensional quasi-projective variety called the *period space of K3 surfaces*. The point defined by $\phi_{\mathbb{C}}(H^{2,0}(X))$ is the *period point of the marked K3 surface* (X, ϕ).

The Weak Torelli theorem gives that two K3 surfaces are isomorphic if and only if there are markings for them such that the corresponding period points are the same. Combining this with the following theorems

Theorem 3 (Local Torelli)**.** *The map from the versal deformation space of X to Ω_{K3} is a local isomorphism.*

Theorem 4 (Surjectivity of the period map)**.** *Every point of Ω_{K3} occurs as the period point of some marked K3 surface.*

We seem to be close to having a moduli space for K3 surfaces: all that remains is to quotient Ω_{K3} by the action of a group to identify period points corresponding to isomorphic K3's. However, on closer inspection we find that this group action is not nice (it is not properly discontinuous), so the quotient will have bad properties (it won't be Hausdorff).

3 Polarisations

To solve this problem, we will restrict our attention to a subclass of K3 surfaces that have better properties: the *polarised* K3 surfaces.

Definition 5. A *(pseudo-)polarised K3 surface of degree* $2k$ (for $k > 0$) is a pair (X, h) consisting of a K3 surface X and a (pseudo-)ample class $h \in H^2(X, \mathbb{Z})$ with $h.h = 2k$.

For pseudo-polarised K3 surfaces we have an upgraded version of the Torelli theorem, which will enable us to build a moduli space for them.

Theorem 6 (Strong Torelli)**.** *Let (X, h) and (X', h') be pseudo-polarised K3 surfaces. Assume that there is a Hodge isometry $\varphi\colon H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ with $\varphi(h') = h$. Then there is a* unique *isomorphism $f\colon X \to X'$ with $\varphi = f^*$.*

We now construct a period space for polarised K3 surfaces. Fix once and for all a primitive class $h \in \Lambda_{\mathrm{K3}}$ with $h^2 = 2k > 0$. Then a *marked (pseudo-) polarised K3 surface of degree* $2k$ is a marked K3 surface (X, ϕ) such that $\phi^{-1}(h)$ is the class of a (pseudo-)ample line bundle on X.

If (X, ϕ) is a marked pseudo-polarised K3 surface of degree $2k$ and if $\omega \in H^{2,0}(X)$ is any class, then $(\omega, \omega) = 0$, $(\omega, \bar{\omega}) > 0$ and $(\omega, h) = 0$. So the period point of (X, ϕ) lies in

$$\Omega_{2k} := \{[\omega] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0, (\omega, h) = 0\}.$$

Ω_{2k} is called the *period space of pseudo-polarised K3 surfaces of degree* $2k$. It is a 19-dimensional quasi-projective variety.

Let $\Gamma(h)$ denote the group of isometries of Λ_{K3} that fix the class h. Then $\Gamma(h)$ acts properly discontinuously on Ω_{2k}, so the quotient $\Gamma(h) \setminus \Omega_{2k}$ will be nicely defined. Strong Torelli and surjectivity of the period map give:

Theorem 7. *The quotient*

$$\mathcal{F}_{2k} := \Gamma(h) \setminus \Omega_{2k}$$

is the moduli space of pseudo-polarised K3 surfaces of degree $2k$.

$\mathcal{F}_{2k}$ is a 19-dimensional quasi-projective variety with only finite quotient singularities. It may be realised as a quotient of a bounded symmetric domain of type IV by an arithmetically defined discrete group of automorphisms, a fact that makes it very amenable to explicit study.

4 Degenerations

For the remainder of this talk, we will discuss what happens when we proceed to the boundary of this moduli space. In order to do this we study *degenerations*.

Definition 8. A *degeneration* of K3 surfaces is a proper, flat, surjective morphism $\pi \colon \mathcal{X} \to \Delta$ from a smooth threefold $\mathcal{X}$ to the unit disc Δ, whose general fibre $X_t = \pi^{-1}(t)$ for $t \neq 0$ is a smooth K3 surface. Note that we do not assume that $\mathcal{X}$ is algebraic, but we will make the assumption that the components of the central fibre $X_0 = \pi^{-1}(0)$ are Kähler.

After a base change and a birational modification, we may always arrange for our degeneration to be *semistable*, i.e. X_0 is a reduced divisor with normal crossings. Furthermore, after an additional birational modification we may arrange that the canonical bundle $\omega_{\mathcal{X}}$ is trivial. The resulting degeneration is called a *Kulikov model*. Kulikov models are useful because there exists a coarse classification for their central fibres.

Theorem 9. *Let* $\pi \colon \mathcal{X} \to \Delta$ *be a Kulikov degeneration. Then either*

(*Type I*) X_0 *is a smooth K3 surface;*

(*Type II*) *X_0 is a chain of elliptic ruled components with rational surfaces at each end, and all double curves are smooth elliptic curves;*

(*Type III*) *X_0 consists of rational surfaces meeting along rational curves which form cycles in each component. If Γ is the dual graph of X_0, then Γ is a triangulation of the 2-sphere.*

These cases can also be distinguished by the action of monodromy on the second cohomology $H^2(X_t, \mathbb{Z})$ of a general fibre. Let T denote the Picard-Lefschetz transformation on $H^2(X_t, \mathbb{Z})$ obtained by transporting classes around 0 and let $N = \log T$. Then N is nilpotent and has $N = 0$ if X_0 is Type I, $N^2 = 0$ and $N \neq 0$ if X_0 is Type II, and $N^3 = 0$ and $N^2 \neq 0$ if X_0 is Type III.

5 Compactifications

Given that we have such a good description of the moduli space for pseudo-polarised K3 surfaces, it is natural to ask whether there is a nice way to compactify this moduli space, i.e. find a compact variety $\overline{\mathcal{F}}_{2k}$ that contains $\mathcal{F}_{2k}$ as an open subset. Ideally, one would like to do this in such a way that the boundary $\overline{\mathcal{F}}_{2k} - \mathcal{F}_{2k}$ encodes some geometric data about the corresponding degenerate fibres.

Probably the best known compactification of $\mathcal{F}_{2k}$ is the Baily-Borel-Satake compactification. This is a method to compactify any arithmetic quotient of a bounded symmetric domain, of which $\mathcal{F}_{2k}$ is an example. In this case the boundary is a union of 0- and 1- dimensional strata. These have some geometric meaning: the 0-dimensional strata correspond to degenerate fibres of Type III, and the 1-dimensional strata to fibres of Type II. Furthermore, the 1-dimensional strata are all rational curves, which are parametrised by the j-invariant of the elliptic double curves of the corresponding Type II degeneration.

Unfortunately this is about all one can say: the boundary in the Baily-Borel-Satake compactification is simply too small to encode more detailed geometric data about degenerate fibres. Many other compactifications exist (toroidal, GIT, KSBA), encapsulating varying amounts of geometry, but as yet there does not seem to be a canonical choice amongst them.

References

[BHPvdV04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. van de Ven, *Compact complex surfaces*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Fogle, A Series of Modern Surveys in Mathematics, vol. 4, Springer-Verlag, 2004.

[Sca87] F. Scattone, *On the compactification of moduli spaces for algebraic K3 surfaces*, Mem. Amer. Math. Soc. **70** (1987), no. 374.