Models For Ample $\langle 2 \rangle$-Polarised K3-Fibrations

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This talk is based upon my current work on an explicit construction for the relative canonical model of an ample $\langle 2 \rangle$-polarised K3-Fibration. When this work is substantially complete I will make it available as a preprint on my website:

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1 Basic Definitions

A *K3 surface* is a nonsingular 2-dimensional compact complex manifold X satisfying

- The canonical class $K_X \sim 0$, and
- The first Betti number $h^1(X, \mathcal{O}_X) = 0$.

We say that a K3 surface is *ample* $\langle 2 \rangle$*-polarised* if there exists a divisor $\mathcal{L}$ on X such that

- $\mathcal{L}$ is ample, and
- $\mathcal{L}^2 = 2$.

Let S be a nonsingular complex curve. We wish to study fibrations $\pi : X \to S$, where X is a threefold (that is not necessarily algebraic) admitting a line bundle $\mathcal{L}$, and the general fibre of π is a K3 surface that is endowed with an ample $\langle 2 \rangle$-polarisation by the restriction of $\mathcal{L}$. Specifically, we wish to construct explicit projective models for such fibrations.

2 Examples

Before we begin, however, it will be useful to know what an ample $\langle 2\rangle$-polarised K3 surface looks like.

Example 1 Let X be a K3 surface, and let $\mathcal{L}$ be an ample divisor on X satisfying $\mathcal{L}^2 = 2$. By results of Mayer [May72], $\mathcal{L}$ is generated by its global sections and $\mathcal{L}^{\otimes 3}$ is very ample. Furthermore, by an easy application of Riemann-Roch

$$h^0(X, \mathcal{L}^{\otimes n}) = n^2 + 2,$$

so $\mathcal{L}$ defines an isomorphism

$$X \cong X_6 \subset \mathbb{P}(1, 1, 1, 3).$$

Note that this implies that X admits a double cover $X \to \mathbb{P}^2$ ramified over a smooth sextic curve.

Example 2 It will be useful to see what happens if we relax the assumptions on $\mathcal{L}$. Rather than assume that $\mathcal{L}$ is ample, we assume instead that it is nef and big (e.g. $\mathcal{L}.C \geq 0$ for all irreducible curves C and $\mathcal{L}^2 > 0$). Then we still have

$$h^0(X, \mathcal{L}^{\otimes n}) = n^2 + 2,$$

but $\mathcal{L}$ is not necessarily generated by its global sections. If this is the case, by results of Mayer [May72], a general member of the linear system defined by $\mathcal{L}$ looks like $(F + 2E)$, where F is the fixed part (a bunch of rational (-2)-curves) and E is elliptic. Then $\mathcal{L}$ defines a morphism

$$X \to X_{2,6} \subset \mathbb{P}(1, 1, 1, 2, 3),$$

where the degree 2 relation does not include the degree 2 variable. This morphism contracts F to the singular point $(0 : 0 : 0 : 1 : a) \in X_{2,6}$. In this case, $X_{2,6}$ cannot be seen as a double cover of $\mathbb{P}^2$. Instead it is a double cover of the cone $X_2 \subset \mathbb{P}(1, 1, 1, 2)$, ramified over a smooth sextic and the vertex $(0 : 0 : 0 : 1)$.

Fibres of the second type will turn out to be very important later.

3 Elliptic Curves

We now want to start studying ample $\langle 2\rangle$-polarised K3-fibrations. As a K3 surface can be seen as a "2-dimensional elliptic curve", it makes sense to draw inspiration from the well-developed theory of elliptic fibrations.

We know that any elliptic curve (in $\mathbb{A}^2$) can be written in Weierstrass form as

$$y^2 = x^3 + Ax + B.$$

We can complete this to a curve in $\mathbb{P}(1,2,3)$ given by

$$Y^2 = X^3 + AXZ^4 + BZ^6.$$

This is, in fact, the equation of a general sextic $X_6 \subset \mathbb{P}(1,2,3)$.

Given an elliptic fibration $X \to S$ admitting a section, Nakayama [Nak88] shows that we can construct a model W for X as a sextic divisor in a $\mathbb{P}(1,2,3)$-bundle over S. Furthermore, there is a morphism $X \to W$ that contracts exactly those components of fibres in X that do not meet the section. This is the *Weierstrass model* of an elliptic fibration.

4 K3-Weierstrass Models

We will attempt to emulate this construction for an ample $\langle 2 \rangle$-polarised K3-fibration $\pi : X \to S$. In order to do this, we construct a $\mathbb{P}(1,1,1,3)$-bundle over S and attempt to map X into it.

To construct this bundle, we start with a rank 3 vector bundle $\mathcal{E}_1$ and a line bundle $\mathcal{E}_3$ on S, then take the **Proj** of the weighted symmetric algebra that has $\mathcal{E}_1$ in degree 1 and $\mathcal{E}_3$ in degree 3. But how do we define $\mathcal{E}_1$ and $\mathcal{E}_3$?

$\mathcal{E}_1$ is easily defined, we simply set $\mathcal{E}_1 := \pi_*\mathcal{L}$. We would like to set $\mathcal{E}_3$ to be the cokernel of the map $\mathrm{Sym}^3(\mathcal{E}_1) \to \pi_*\mathcal{L}^{\otimes 3}$ but, due to the existence of the $\mathbb{P}(1,1,1,2,3)$-fibres, this cokernel is not locally free. Instead, we are forced to define $\mathcal{E}_3$ as the reflexivisation

$$\mathcal{E}_3 := \left(\mathrm{coker}\left(\mathrm{Sym}^3(\mathcal{E}_1) \to \pi_*\mathcal{L}^{\otimes 3}\right)\right)^{\vee\vee}.$$

Finally, we take a sextic hyperplane section in this bundle to get a model W of X. There is a birational map $\mu : X - \to W$ that is an isomorphism on the general fibre. But what happens to the special fibres?

It can be shown that, on a $\mathbb{P}(1,1,1,2,3)$-fibre, μ is the natural projection $\mathbb{P}(1,1,1,2,3) - \to \mathbb{P}(1,1,1,3)$. Unfortunately this projection destroys the structure of these fibres, and their images in $\mathbb{P}(1,1,1,3)$ are potentially very singular. These singularities make the K3-Weierstrass model very difficult to study.

5 Genus Two Curves

To solve this problem, we turn our attention away from elliptic curves, and instead look at the theory of fibrations by genus two curves. Whilst this seems like a strange idea, the general genus two curve is a double cover of $\mathbb{P}^1$ ramified over six points, i.e. a sextic in $\mathbb{P}(1,1,3)$. This is closely analogous to our situation.

Horikawa [Hor77] attempts to construct a model for fibrations by genus two curves by constructing a $\mathbb{P}^1$-bundle on the base, then taking a double cover ramified over a sextic divisor. This is, in fact, analogous to what we did with the K3-Weierstrass model in the last section. However, he encounters the same problem that we did: there exist genus two curves (called unigonal curves) that are not double covers of $\mathbb{P}^1$. Instead, they are double covers of the quadric cone $X_2 \subset \mathbb{P}(1,1,2)$. Exactly as in our case, the structure of these curves is destroyed when he proceeds to his model, and this leads to unmanageable singularities in the total space.

Fortunately, Catanese and Pignatelli [CP06] find a way to solve this problem. Rather than constructing their model as a double cover of a $\mathbb{P}^1$-bundle, they use a double cover of a conic bundle instead. Then the general fibres are still isomorphic to $\mathbb{P}^1$, but these fibres can now degenerate to quadric cones. This allows the structure of the unigonal fibres to be preserved.

So how does this construction work? Given a genus two fibration $\pi : X \to S$, Catanese and Pignatelli find a way to explicitly construct the relative canonical model

$$X^c := \mathbf{Proj}\left(\bigoplus_{n=0}^{\infty} \pi_* K_X^{\otimes n}\right)$$

as a double cover of a certain conic bundle. Furthermore, this construction only depends upon a relatively small set of input data. Finally, by the properties of the relative canonical model (see [KM98, Section 3.8]), there is a birational morphism $\phi : X \to X^c$ contacting only those curves C that are contained in a fibre and have $K_X.C = 0$, and X^c has at worst canonical singularities.

6 Relative Canonical Models

We would like to adapt Catanese and Pignatelli's [CP06] method to work for an ample $\langle 2\rangle$-polarised K3-fibration $\pi : X \to S$. However, we immediately encounter a problem: as defined above, the relative canonical model does not carry much information, since K_X vanishes on the fibres of π.

Instead, we construct the relative canonical model of the log pair $(X, \mathcal{L})$. This is defined by

$$X^c := \mathbf{Proj}\left(\bigoplus_{n=0}^{\infty} \pi_* K_X^{\otimes n} \otimes \pi_* \mathcal{L}^{\otimes n}\right).$$

Then we know that, under certain (semistability) assumptions on X, there is a birational morphism $\phi : X \to X^c$ and X^c has at worst canonical singularities.

In this situation, X^c is an excellent projective model for X. Furthermore, it can be explicitly constructed as a double cover of a rational surface bundle, using only a reasonably small set of input data.

References

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