Models for Threefolds Fibred by K3 Surfaces of Degree Two

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This talk is based upon my current work on an explicit construction for the relative log canonical model of a threefold fibred by K3 surfaces of degree two. When this work is substantially complete I will make it available as a preprint on my website:

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1 Basic Definitions

A *K3 surface* is a nonsingular 2-dimensional compact complex manifold X satisfying

- The canonical sheaf $\omega_X \cong \mathcal{O}_X$, and
- The first Betti number $h^1(X, \mathcal{O}_X) = 0$.

If X is a K3 surface, a line bundle $\mathcal{L}$ on X is called a *polarisation of degree two* if

- $\mathcal{L}$ is ample, and
- $\mathcal{L}.\mathcal{L} = 2$.

If such an $\mathcal{L}$ exists, X is called a *K3 surface of degree two.*

Let S be a nonsingular complex curve. We wish to study fibrations (i.e. proper, flat, surjective morphisms) $\pi : X \to S$, where X is a nonsingular threefold that admits a line bundle $\mathcal{L}$ such that the general fibre of π is a K3 surface that is endowed with a polarisation of degree two by the restriction of $\mathcal{L}$. Specifically, we wish to construct explicit projective models for such fibrations.

Before we begin, however, it will be useful to know what a K3 surface of degree two looks like.

Example 1 Let X be a K3 surface, and let $\mathcal{L}$ be an ample line bundle on X satisfying $\mathcal{L}.\mathcal{L} = 2$. By results of Mayer [May72], $\mathcal{L}$ is generated by its global sections and $\mathcal{L}^{\otimes 3}$ is very ample. Furthermore, by an easy application of Riemann-Roch

$$h^0(X, \mathcal{L}^{\otimes n}) = n^2 + 2,$$

so $\mathcal{L}$ defines an isomorphism

$$X \cong X_6 \subset \mathbb{P}(1,1,1,3).$$

Note that this implies that X admits a double cover $X \to \mathbb{P}^2$ ramified over a smooth sextic curve.

2 Elliptic Curves

We now want to start studying threefolds fibred by K3 surfaces of degree two. As a K3 surface can be seen as a "2-dimensional elliptic curve", it makes sense to draw inspiration from the well-developed theory of elliptic fibrations.

We know that any elliptic curve (in $\mathbb{A}^2$) can be written in Weierstrass form as

$$y^2 = x^3 + Ax + B.$$

We can complete this to a curve in $\mathbb{P}(1,2,3)$ given by

$$Y^2 = X^3 + AXZ^4 + BZ^6.$$

This is, in fact, the equation of a general sextic $X_6 \subset \mathbb{P}(1,2,3)$.

Given an elliptic fibration $X \to S$ admitting a section, Nakayama [Nak88] shows that we can construct a model W for X as a sextic hypersurface in a $\mathbb{P}(1,2,3)$-bundle over S. Furthermore, there is a morphism $X \to W$ that contracts exactly those components of fibres in X that do not meet the section. This is the *Weierstrass model* of an elliptic fibration.

3 K3-Weierstrass Models

We will attempt to emulate this construction for a threefold fibred by K3 surfaces of degree two $\pi : X \to S$. In order to do this, we construct a $\mathbb{P}(1,1,1,3)$-bundle over S and attempt to map X into it.

To construct this bundle, we start with a rank 3 vector bundle $\mathcal{E}_1$ and a line bundle $\mathcal{E}_3$ on S, then take the **Proj** of the weighted symmetric algebra that has $\mathcal{E}_1$ in degree 1 and $\mathcal{E}_3$ in degree 3. But how do we define $\mathcal{E}_1$ and $\mathcal{E}_3$?

$\mathcal{E}_1$ is easily defined, we simply set $\mathcal{E}_1 := \pi_*\mathcal{L}$. We would like to set $\mathcal{E}_3$ to be the cokernel of the map $\mathrm{Sym}^3(\mathcal{E}_1) \to \pi_*\mathcal{L}^{\otimes 3}$ but unfortunately this cokernel is not locally free. Instead, we are forced to define $\mathcal{E}_3$ as the reflexivisation

$$\mathcal{E}_3 := \left(\mathrm{coker}\left(\mathrm{Sym}^3(\mathcal{E}_1) \to \pi_*\mathcal{L}^{\otimes 3}\right)\right)^{\vee\vee}.$$

Finally, we take a sextic hypersurface section in this bundle to get a model W of X.

Theorem 2 *There is a birational map $\mu : X - \to W$ over S that is an isomorphism on the general fibre.*

Note that this theorem tells us almost nothing about the special fibres. To see what happens to them, we need to understand what such fibres look like.

Example 3 We can obtain one type of special fibre by relaxing the assumptions on $\mathcal{L}$. Rather than assume that $\mathcal{L}$ is ample, we assume instead that it is nef and big (e.g. $\mathcal{L}.C \geq 0$ for all irreducible curves C and $\mathcal{L}.\mathcal{L} > 0$). Then we still have

$$h^0(X, \mathcal{L}^{\otimes n}) = n^2 + 2,$$

but $\mathcal{L}$ is not necessarily generated by its global sections. If this is the case, by results of Mayer [May72], a general member of the linear system defined by $\mathcal{L}$ looks like $(F + 2E)$, where F is the fixed part (a bunch of rational (-2)-curves) and E is elliptic. Then $\mathcal{L}$ defines a morphism to a complete intersection

$$X \to X_{2,6} \subset \mathbb{P}(1,1,1,2,3),$$

where the degree 2 relation does not include the degree 2 variable. This morphism contracts F to the singular point $(0:0:0:1:a) \in X_{2,6}$. In this case, $X_{2,6}$ cannot be seen as a double cover of $\mathbb{P}^2$. Instead it is a double cover of the cone $X_2 \subset \mathbb{P}(1,1,1,2)$, ramified over a smooth sextic and the vertex $(0:0:0:1)$.

In fact it can be shown that, under a semistability assumption on the fibres of $\pi : X \to S$, we have:

Theorem 4 *If all fibres of $\pi : X \to S$ are semistable, then every fibre admits a morphism to either a (possibly singular) sextic hypersurface $X_6 \subset \mathbb{P}(1,1,1,3)$ or to a complete intersection $X_{2,6} \subset \mathbb{P}(1,1,1,2,3)$, where the degree 2 relation does not include the degree 2 variable.*

It can be shown that, on a $\mathbb{P}(1,1,1,2,3)$-fibre, μ is the natural projection $\mathbb{P}(1,1,1,2,3) \dashrightarrow \mathbb{P}(1,1,1,3)$. Unfortunately this projection destroys the structure of these fibres, and their images in $\mathbb{P}(1,1,1,3)$ are potentially very singular. These singularities make the K3-Weierstrass model very difficult to study.

4 Genus Two Curves

To solve this problem, we turn our attention away from elliptic curves, and instead look at the theory of fibrations by genus two curves. Whilst this seems like a strange idea, the general genus two curve is a double cover of $\mathbb{P}^1$ ramified over six points, i.e. a sextic in $\mathbb{P}(1,1,3)$. This is closely analogous to our situation.

Horikawa [Hor77] attempts to construct a model for fibrations by genus two curves by constructing a $\mathbb{P}^1$-bundle on the base, then taking a double cover ramified over a sextic divisor. This is, in fact, analogous to what we did with the K3-Weierstrass model in the last section. However, he encounters the same problem that we did: there exist genus two curves (called unigonal curves) that are not double covers of $\mathbb{P}^1$. Instead, they are double covers of the quadric cone $X_2 \subset \mathbb{P}(1,1,2)$. Exactly as in our case, the structure of these curves is destroyed when he proceeds to his model, and this leads to unmanageable singularities in the total space.

Fortunately, Catanese and Pignatelli [CP06] find a way to solve this problem. Rather than constructing their model as a double cover of a $\mathbb{P}^1$-bundle, they use a double cover of a conic bundle instead. Then the general fibres are still isomorphic to $\mathbb{P}^1$, but these fibres can now degenerate to quadric cones. This allows the structure of the unigonal fibres to be preserved.

So how does this construction work? Given a genus two fibration $\pi : X \to S$, Catanese and Pignatelli find a way to explicitly construct the relative canonical model

$$X^c := \mathbf{Proj}_S \left(\bigoplus_{n=0}^{\infty} \pi_* \omega_X^{\otimes n} \right)$$

as a double cover of a certain conic bundle. Furthermore, this construction only depends upon a relatively small set of input data, that is easily determined from the fibration $\pi : X \to S$. Finally, by the properties of the relative canonical model (see [KM98, Section 3.8]), there is a birational morphism $\phi : X \to X^c$ contacting only those curves C that are contained in a fibre and have $\omega_X . C = 0$, and X^c has at worst canonical singularities.

5 Relative Canonical Models

We would like to adapt Catanese and Pignatelli's [CP06] method to work for a threefold fibred by K3 surfaces of degree two $\pi : X \to S$. However, we immediately encounter a problem: as defined above, the relative canonical model does not carry much information, since ω_X vanishes on the fibres of π.

Instead, we construct the relative log canonical model of the log pair $(X, \mathcal{L})$. This is defined by

$$X^c := \mathbf{Proj}_S \left(\bigoplus_{n=0}^{\infty} \pi_* \omega_X^{\otimes n} \otimes \pi_* \mathcal{L}^{\otimes n} \right).$$

We have:

Theorem 5 *If the fibres of $\pi : X \to S$ are semistable, there is a birational morphism $\phi : X \to X^c$ over S and X^c has at worst canonical singularities. Furthermore, X^c may be constructed explicitly from a certain set of data on S, all of which is easily determined from the fibration $\pi : X \to S$.*

References

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