Degenerations Of ⟨2⟩-Polarised K3 Surfaces

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This talk is based upon the preprint *Degenerations of ⟨2⟩-Polarised K3 Surfaces*, which is available from the preprints section of my website:

http://people.maths.ox.ac.uk/∼thompsona

Full proofs of all results (or references to them) may be found there.

This talk follows on from two talks (*The Construction of Ample ⟨2⟩-Polarised K3-Fibrations* and *Degenerations of Surfaces with Kodaira Number Zero*) given last year. However, I don't expect people to remember the details, so we will recap them briefly now.

Definition. *A* ⟨2⟩-Polarised K3 Surface (X, H) *is a 2-dimensional nonsingular complex algebraic surface* X *with* $K_X \sim 0$ *and* $b_1(X) = 0$ *(a* K3 surface*), and an ample divisor* H *on* X *satisfying* $H^2 = 2$

If H is base point free, H defines an isomorphism

$$\begin{array}{ccc} X \xrightarrow{\sim} X_6 = \{z^2 = f_6(x_1,x_2,x_3)\} & \subset & \mathbb{P}_{(1,1,1,3)}[x_1,x_2,x_3,z] \\ \downarrow{\scriptstyle 2:1} & & \\ \mathbb{P}^2[x_1,x_2,x_3] & & \end{array}$$

exhibiting X as a double cover of $\mathbb{P}^2$ ramified over a nonsingular sextic curve $f_6(x_1, x_2, x_3) = 0$. H is mapped to the pull-back of the hyperplane section in $\mathbb{P}^2$ under this map. We are interested in studying families of such things.

Let Δ denote a small complex disc. Let $\pi : Y \to \Delta$ be a projective, flat, surjective morphism of normal complex varieties whose general fibres (i.e. fibres over $\Delta^* = \Delta \setminus \{0\}$) are K3 surfaces. Let H be a divisor on Y that induces a base point free ⟨2⟩-polarisation on a general fibre.

Then it can be shown that there exists a birational map

$$\mu : Y \dashrightarrow W \subset \mathbb{P}(1, 1, 1, 3) \times \Delta$$

that is an isomorphism over Δ^*, and such that the morphism $Y_t \to \mathbb{P}(1,1,1,3)$ induced on a general fibre agrees with the morphism defined on Y_t by the polarisation induced by H (that was described above). We want to know what happens to the fibre of Y over 0 when we apply μ to it.

However, in order to be able to do this we will need to make some assumptions on Y. We will assume that:

- Y is semistable, i.e. Y is nonsingular and $Y_0 = \pi^{-1}(0)$ is a reduced divisor with normal crossings. By a theorem of Knudsen-Mumford-Waterman ('73), this can always be arranged by a composition of a base change and a series of birational modifications.

- Y is Kulikov, i.e. $K_Y \sim 0$. By a theorem of Kulikov ('77, '81) and Persson-Pinkham ('81), this can always be arranged by a further series of birational modifications.

The central fibres in such Y have been classified by Persson ('77), Kulikov ('77) and Friedman-Morrison ('83). We make use of this classification when finding the images of these fibres under μ.

So what can the central fibres Y_0 be? There are 4 ways in which the assumptions on the general fibre may fail on Y_0:

(I) Y_0 is a smooth K3 surface, H_0 (the divisor induced on Y_0 by H) is base point free but not ample.

(II) Y_0 is not smooth, but a "Type II" degeneration of K3 surfaces, in the classification mentioned above.

(III) Y_0 is not smooth, but a "Type III" degeneration of K3 surfaces, in the classification mentioned above.

(IV) Y_0 is a smooth K3 surface, and H_0 has base points.

We will focus on case (II) in this talk, and see what the effect of μ is on a fibre of this type.

In order to do this, first we need to know what a fibre of Type II looks like. By the classification mentioned before, the central fibre Y_0 in a Type II degeneration of K3 surfaces is a chain $V_0 \cup \cdots \cup V_r$ of surfaces such that:

- $V_i \cap V_{i+1} = D_i$ is a smooth elliptic curve for all i.

- $V_i \cap V_j = \emptyset$ if $|i - j| > 1$.

- $V_1, \ldots, V_{r-1}$ are surfaces ruled over $D_0, \ldots, D_{r-2}$ respectively.

- V_0 and V_r are rational surfaces, i.e. birational to $\mathbb{P}^2$.

Now we have this, we can proceed with finding the image of such a fibre under μ. In order to do this, we aim to find a "special" divisor H' on Y, equal to H over Δ^*, such that the linear system induced by H' on *every* fibre of Y defines a birational map to $\mathbb{P}(1,1,1,3)$. Then, by the properness of the Hilbert scheme, H' will define the map μ on Y, and the induced divisor H'_0 on Y_0 will define the restriction of μ to Y_0. We have the diagram

$$\begin{array}{ccc} Y' & \overset{H'}{\dashrightarrow} & W \subset \mathbb{P}(1,1,1,3) \times \Delta \\ \uparrow & & \uparrow \\ Y_0 & \overset{H'_0}{\dashrightarrow} & W_0 \subset \mathbb{P}(1,1,1,3) \end{array}$$

where our aim is to find W_0. In order to do this, however, we need to know what this "special" H' is. We have:

Theorem (Friedman '84)**.** *There exist integers a_i such that, after a series of birational modifications have been applied to Y, the twisted divisor $H' = H + \sum_{i=0}^{r} a_i V_i$ satisfies:*

1. *H' is numerically effective.*
2. *$V_1, \ldots, V_{r-1}$ are minimal ruled.*
3. *$H'|_{V_i}$ is a sum of fibres for $1 \leq i \leq r-1$.*
4. *$H'.D_i > 0$ for all i.*

This H' turns out to be exactly what we are looking for.

Let μ_0 be the map defined on Y_0 by H'_0 (which agrees with the restriction of μ to Y_0). Then properties (2)-(4) above imply that μ_0 contracts $V_1, \ldots, V_{r-1}$ onto $D_0 \cong D_{r-1}$. This leaves us with two rational surfaces V_0 and V_r meeting along an elliptic curve $D := D_0$.

Let $H'_0 := H'|_{V_0}$ and $H'_r := H'|_{V_r}$. Then $(H'_0)^2 + (H'_r)^2 = 2$ and, by (1) in the theorem, $(H'_i)^2 \geq 0$ for each i. By Riemann-Roch and adjunction, there are four possibilities:

1. H'_i is connected for $i = 0, r$ with $g(H'_0) = g(H'_r) = 0$, $(H'_0)^2 = (H'_r)^2 = 1$ and $H'_i.D = 3$.
2. H'_i is connected for $i = 0, r$ with $g(H'_0) = g(H'_r) = 1$, $(H'_0)^2 = (H'_r)^2 = 1$ and $H'_i.D = 1$.

3. H_i' is connected for $i = 0, r$ with $g(H_0') = 1$, $g(H_r') = 0$, $(H_0')^2 = 2$, $(H_r')^2 = 0$ and $H_i'.D = 2$.

4. H_0' is connected but $|H_r'|$ has no connected members. $g(H_0') = 0$, $(H_0')^2 = 2$, $(H_1')^2 = 0$ and $H_i'.D = 4$.

We will show by means of an example how to use this information to calculate the image of μ_0.

Consider case (3) above. We have rational surfaces V_0 and V_r meeting along a smooth elliptic curve D. There are connected divisors H_0' on V_0 and H_r' on V_r with $g(H_0') = 1$, $g(H_r') = 0$, $(H_0')^2 = 2$, $(H_r')^2 = 0$ and $H_i'.D = 2$.

By the properties of rational surfaces, we may assume that the linear system $|H_0'|$ has no fixed components or base points, and contains irreducible members. This means that the restriction of μ_0 to V_0 will be a morphism (i.e. no points will be blown up).

By Riemann-Roch, $|H_0'|$ defines a morphism

$$\varphi_{H_0} : V_0 \longrightarrow \{z^2 = f_4(x_i)\} \subset \mathbb{P}_{(1,1,1,2)}[x_1, x_2, x_3, z],$$

where $\{f_4(x_i) = 0\}$ is a quartic in $\mathbb{P}^2$ with at worst A-D-E singularities. D is mapped to the nonsingular curve $\{l(x_i) = z^2 - f_4(x_i) = 0\}$, where l is linear. This is a double cover of the line $\{l(x_i) = 0\} \subset \mathbb{P}^2$ ramified over four points (i.e. an elliptic curve).

Next we look at V_r. Again, $|H_r'|$ has no fixed components or base points. This time, by Riemann-Roch, $|H_r'|$ defines a morphism

$$\varphi_{H_r} : V_r \longrightarrow \mathbb{P}^1.$$

contracting V_r to $\mathbb{P}^1$. Under this morphism D is mapped surjectively to $\mathbb{P}^1$, and the restriction of φ_{H_r} exhibits D as a double cover of $\mathbb{P}^1$ ramified over four points.

The contraction of V_r is realised on V_0 by the map:

$$\begin{aligned} g : \{z^2 - f_4(x_i) = 0\} &\longrightarrow \{w^2 - l^2(x_i) f_4(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, w] \\ (x_i, z) &\longmapsto (x_i, l(x_i) z) \end{aligned}$$

So the map defined on V_0 by the restriction of H is given by $\varphi_H(V_0) = g \circ \varphi_{H_0}(V_0)$, and

$$W_0 = \{w^2 - l^2(x_i) f_4(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, w].$$