An Introduction To The Birational Classification Of Surfaces

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References

(B) - A. Beauville - Complex Algebraic Surfaces
(R) - M. Reid - Chapters On Algebraic Surfaces
(W) - P. Wilson - Towards Birational Classification Of Algebraic Varieties
(BPV) - Barth, Peters, van de Ven - Compact Complex Surfaces

1 Introduction

Birational equivalence is very much the natural notion of equivalence to use when studying varieties in algebraic geometry. The topological notions of homeomorphism and diffeomorphism are too flexible to preserve algebro-geometric structure, and isomorphism proves too rigid. In the words of Miles Reid, birational equivalence says that "the meat of the varieties is the same, although they may differ a bit around the edges".

In this talk, a *surface* will refer to a smooth, projective variety of dimension 2 over the complex numbers (note: Projective implies connected and irreducible).

2 What is birational equivalence?

In this section, X and Y denote varieties with X irreducible.

Definition 2.1 *(B II.4) A* rational map $\phi : X - \to Y$ *is a morphism from a (Zariski) open subset $U \subset X$ to Y which cannot be extended to a larger open subset.*

(Note: A morphism is roughly a continuous map that is everywhere regular - here regularity is defined by composing ϕ with any regular map $f : Y \to \mathbb{C}$, and seeing if this composition defines a regular map. To define morphisms properly we need the mechanics of schemes, which will not be covered here, see *Hartshorne - Algebraic Geometry*).

Definition 2.2 *A* birational map $\phi : X - \to Y$ *is a rational map with rational inverse.*

Unsurprisingly, two varieties are called *birationally equivalent* if there exists a birational map between them.

3 The plan of action

There are two sides to the classification:

1) **The calculation of birational invariants** This allows us to divide birational equivalence classes of surfaces into a number of broad "families", with similar properties. It is quite a coarse means of classification.

2) **The construction of minimal models** This attempts to find a "special" representative from each birational equivalence class of surfaces, which can then be studied. It is a very fine means of classification.

To be difficult, we'll start by studying the minimal model side of the classification:

4 Minimal models

From this point onwards, S will denote a surface.

Denote by $B(S)$ the set of isomorphism classes of surfaces birationally equivalent to S. $B(S)$ is ordered by *domination*: We say that $\hat{S}$ dominates S if there exists a birational morphism $\phi : \hat{S} \to S$ (i.e. ϕ is defined at all points of S).

Definition 4.1 *(B II.15) A surface S is* minimal *if its class in $B(S)$ is minimal with respect to the domination ordering (i.e. Every birational morphism $\phi : S \to S'$ is an isomorphism).*

Our main tool in the construction of minimal models is the operation of blowing-up:

Definition 4.2 *(B II.1) If S is a surface and $p \in S$, then there exists a surface $\hat{S}$ and a morphism $\epsilon : \hat{S} \to S$, unique up to isomorphism, such that:*
a) The restriction of ϵ to $\epsilon^{-1}(S - \{p\})$ is an isomorphism onto $S - \{p\}$.
b) $E = \epsilon^{-1}$ is isomorphic to $\mathbb{P}^1$.

Such a curve E is called an *exceptional curve*.

Theorem 4.3 *(B II.11) Let $f : S \to S_0$ be a birational morphism of surfaces. Then there exists a (finite) sequence of blow-ups $\epsilon_k : S_k \to S_{k-1}$ $(1 \leq k \leq n)$ and an isomorphism $u : S \to S_n$ such that $f = \epsilon_1 \circ \epsilon_2 \circ \cdots \circ \epsilon_n \circ u$.*

Importantly, this implies that a surface is minimal if and only if it contains no exceptional curves. This fact, along with the contractibility criterion of Castelnuovo (B II.17) (which characterises exceptional curves on a surface), allows us to find a minimal model for any given surface. This proves that a minimal model for any surface exists.

However, this tells us nothing about whether minimal models are unique! For this, we require a brief foray into ruled surfaces:

5 Ruled surfaces

Definition 5.1 *(B III.1) A surface S is* ruled *if it is birationally equivalent to $C \times \mathbb{P}^1$, for some smooth curve C.*

Unfortunately, an example will show us that minimal models of ruled surfaces are not necessarily unique:

Example 5.2 (B III.24.1) Let S be a minimal ruled surface, $p \in S$ a point of S, and F the fibre of S containing p. Blow-up S at p to obtain a new surface, $\hat{S}$. Denote the strict transform of F by $\hat{F}$. Then $\hat{F}$ is an exceptional curve in $\hat{S}$. Blow-down $\hat{F}$ to obtain a new minimal ruled surface that is not isomorphic to the first.

However, despite this setback, we can still salvage something from the situation:

Theorem 5.3 *(B V.19) Let S, S' be two non-ruled minimal surfaces. Then every birational map $S - \to S'$ is an isomorphism*

i.e. Every non-ruled surface admits a unique minimal model. So, as long as we stick to non-ruled surfaces, we need only classify minimal surfaces up to isomorphism.

Minimal models provide a good solution to the classification problem, but we'd like an easier way to gain useful information about the birational equivalence class of a surface without going through the intricacies of calculating minimal models.

6 Birational invariants

There are a few important numbers that we can define that give us a lot of information about the birational equivalence class of a surface.

(**Note:** In what follows, K denotes the canonical divisor, a divisor satisfying $\mathcal{O}_S(K) \cong \Omega_S^2$)

- $p_g(S) = h^2(\mathcal{O}_S) = h^0(\mathcal{O}_S(K))$, the geometric genus.
- $q(S) = h^1(\mathcal{O}_S)$, the irregularity.
- $P_n(S) = h^0(\mathcal{O}_s(nK))$ $(n \geq 1)$, the nth plurigenus.

However, the plurigenera are generally difficult to work with (there are infinitely many of them!). We can simplify matters by instead considering the Kodaira dimension which, as we shall see, stores a lot of the useful information carried by the plurigenera in a single value.

Definition 6.1 *(B VII.6) The* Kodaira dimension *of a surface S, denoted $\kappa(S)$, is defined as:*

- $\kappa(S) = -\infty \Leftrightarrow P_{12}(S) = 0$
- $\kappa(S) = 0 \Leftrightarrow P_{12}(S) = 1$
- $\kappa(S) = 1 \Leftrightarrow P_{12}(S) \geq 2$ *and* $K^2 = 0$
- $\kappa(S) = 2 \Leftrightarrow P_{12}(S) \geq 2$ *and* $K^2 > 0$

There is a much more general definition of the Kodaira dimension for any smooth projective variety over $\mathbb{C}$, but it is beyond the scope of this talk (B VII.1).

To conclude this talk, we will examine each value taken by the Kodaira dimension in turn, to see what we can deduce about the surfaces that take these values.

$\kappa(S) = -\infty$: This case occurs if and only if S is ruled over some smooth curve C (This follows from Enriques' theorem (B VI.17)). We automatically get the following values for the birational invariants (B III.21):

- $q(S) = g(C)$ (g denotes the usual curve genus)
- $p_g(S) = 0$
- $P_n(S) = 0$ for all $n \geq 1$

$\kappa(S) = 1$: (B IX) In this case, we have a useful theorem, courtesy of Iitaka:

Theorem 6.2 (Iitaka fibration) *(W 1.1) Let V be a smooth projective variety. If $\kappa(V) \geq 0$, there exists $\hat{V}$, birational to V, and a fibration $\varphi : \hat{V} \to W$ over a smooth projective variety W such that* $\dim(W) = \kappa(V)$ *and* $\kappa(\hat{V}_w) = 0$ *for the general fibre $\hat{V}_w$.*

In our case, this says that any surface S with $\kappa(S) = 1$ is birational to a fibration of some smooth curve W by elliptic curves (note: Elliptic curves have $\kappa = 0$). Such a surface is called an *elliptic surface.*

In fact, this theorem is a powerful tool in the classification of general varieties; it effectively reduces the problem of classifying varieties V of dimension n to that of classifying those with $\kappa(V) = -\infty$, 0 or n.

$\kappa(S) = 0$: (B VIII) This is the most interesting case and, in view of Iitaka's fibration, the most useful in higher dimensional classification problems. For this reason, it has been split into several subcases, characterised by the other invariants.

Note that, for any surface with $\kappa(S) = 0$, $P_n(S) = 0$ or 1 for all n, and there exists at least one N for which $P_N(S) = 1$ (B VII.3).

Theorem 6.3 *(B VIII.2) Let S be a minimal surface with $\kappa(S) = 0$. Then S belongs to one of the following four cases:*

- *If $p_g(S) = 0$, $q(S) = 0$, then S is an* Enriques surface
- *If $p_g(S) = 0$, $q(S) = 1$, then S is a* bielliptic surface
- *If $p_g(S) = 1$, $q(S) = 0$, then S is a* K3 surface
- *If $p_g(S) = 1$, $q(S) = 2$, then S is an* abelian surface

All of these cases have been studied in great depth, and there is a large amount known about them.

$\kappa(S) = 2$: (B X) This is the case of *surfaces of general type.* They are the most general of any of the families in the classification, and as such are the hardest to study. Little is known about them in general, save for certain bounds on the values that certain of the birational invariants may take.