Models for Threefolds Fibred by K3 Surfaces of Degree Two

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Abstract

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This thesis focuses on the study of projective models for threefolds that admit fibrations by K3 surfaces of degree two.

In Chapter 1 we construct a K3 analogue of the Weierstrass model of an elliptic fibration, then prove the existence of such a model for any threefold that admits a fibration by K3 surfaces of degree two.

In Chapter 2 we study the relative log canonical model of a variety that admits a fibration over a smooth curve. We prove a condition under which this model exists and use this condition to show that any threefold admitting a semistable fibration by K3 surfaces of degree two has such a model.

In Chapter 3 we study the local form of the relative log canonical model of a threefold fibred by K3 surfaces of degree two. We find an explicit classification for the semistable degenerate fibres occurring in such a model, as complete intersections in certain weighted projective spaces.

In Chapter 4 we show that the relative log canonical model of a semistable threefold fibred by K3 surfaces of degree two can be explicitly reconstructed from a certain easily determined set of data on the base curve. We further prove that, under certain assumptions, any such set of data determines a threefold that arises as the relative log canonical model of some threefold fibred by K3 surfaces of degree two.

In Chapter 5 we explicitly calculate several of the geometric properties of the models that we have found, including the canonical sheaf and coherent Euler characteristic, and give a necessary and sufficient condition for a certain resolution of the models to be Calabi-Yau. We further calculate the rank of the second integral cohomology in the Calabi-Yau case.

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Finally, this thesis is dedicated to my parents. Without their constant support and encouragement I would not be here today.

Statement of Originality

This thesis contains no material that has already been accepted, or is concurrently being submitted, for any degree or diploma or certificate or other qualification in this University or elsewhere. To the best of my knowledge and belief this thesis contains no material previously published or written by another person, except where due reference is made in the text.

Alan Matthew Thompson

3rd March 2011

Contents

0.1 Introduction

This thesis focuses on the study of projective models for threefolds that admit a fibration by K3 surfaces of degree two. The nature of the models considered means that they lend themselves well to explicit study; in particular, it can be shown that they can be completely reconstructed from a relatively small set of starting data which is easily determined from the starting threefold. Moreover, this description allows some of the geometrical properties of the model to be explicitly calculated.

In more detail, a K3 surface is said to have degree two if it admits an ample invertible sheaf with self-intersection number two. Such surfaces have two equivalent descriptions: as double covers of the complex projective plane ramified over smooth sextics, or as sextic hypersurfaces in the weighted projective space $\mathbb{P}(1,1,1,3)$. A nonsingular threefold X is said to admit a K3 fibration if there is a projective, flat and surjective morphism $\pi\colon X \to S$ to a nonsingular curve S, whose general fibres are K3 surfaces. We say that X admits a fibration by K3 surfaces of degree two if, furthermore, there exists an invertible sheaf $\mathcal{L}$ on X (called the *polarisation sheaf*) whose restriction to a general fibre of π is ample and has self-intersection number two. The triple $(X, \pi, \mathcal{L})$ is called a *threefold fibred by K3 surfaces of degree two* over S.

We begin our pursuit of models for such threefolds by looking to the well-developed theory of elliptic fibrations for inspiration. Using the fact that any elliptic curve can be embedded by an equation of Weierstrass form in $\mathbb{P}^2$, Nakayama [Nak88] proves that any complex variety that admits an elliptic fibration with a section has a birational morphism to a projective model, called the *Weierstrass model*. This model is constructed by defining a $\mathbb{P}^2$-bundle over the base variety, then taking a hypersurface in it defined by an equation of Weierstrass form.

Given a threefold fibred by K3 surfaces of degree two $(X, \pi, \mathcal{L})$, we attempt to emulate this construction by defining a $\mathbb{P}(1,1,1,3)$-bundle over our base curve S, then mapping X to a sextic hypersurface within it. Unfortunately there turns out to be a fundamental problem with this method, as a family of K3 surfaces of degree two can degenerate to a so-called *unigonal* surface, that does not admit a morphism to $\mathbb{P}(1,1,1,3)$. Thus any unigonal fibres appearing in X cannot be made to map nicely into our weighted projective bundle. To circumvent this problem we are forced to define our model only on the open set in S over which the fibres embed into $\mathbb{P}(1,1,1,3)$, then to complete across the gaps using the properness of the Hilbert scheme. This construction gives us our first model, the *K3-Weierstrass model*, denoted by W.

The K3-Weierstrass model is simple to construct, depending upon only three pieces of data on the base curve S, all of which are easily determined from $\pi\colon X \to S$ and $\mathcal{L}$. However, the process of "completing across the gaps" means that we cannot guarantee that we will have a morphism $X \to W$; instead we will obtain only a birational map $X - \to W$. Moreover, this birational map destroys the structure of any unigonal fibres of $\pi\colon X \to S$, which makes it difficult to control the singularities appearing in W and thwarts attempts to calculate its properties.

In order to solve this problem, we turn instead to the theory of fibrations by genus two curves for inspiration. It is easy to see the parallel between this and our setup when one notes that a general genus two curve can be seen as a double cover of the projective line ramified over six points, whereas a K3 surface of degree two can be seen as a double cover of the projective plane ramified over a smooth sextic curve.

Indeed, in the case of a fibration $p\colon Z \to S$ of a surface by genus two curves, Horikawa [Hor77] constructs a model that is in many ways analogous to our K3-Weierstrass model, although using a different method to perform the construction itself. After doing so he experiences a similar problem to ours: the map to his model is not necessarily a morphism. This leads to the appearance of highly singular fibres in his models and makes them difficult to study.

Fortunately for us, Catanese and Pignatelli [CP06] find a way to solve Horikawa's problem. Their solution is to find an explicit method to construct the *relative canonical algebra* of Z, defined to be the $\mathcal{O}_S$-algebra

$$\mathcal{R}(Z) := \bigoplus_{n \geq 0} p_*(\omega_{Z/S}^{\otimes n}),$$

in terms of a small set of data on S that is easily determined from the fibration $p\colon Z \to S$. This can then be used to explicitly construct the *relative canonical model* of Z, defined as

$$Z^c := \mathbf{Proj}_S \mathcal{R}(Z).$$

Given this, standard results of the minimal model program give a birational morphism $Z \to Z^c$ and limit the severity of the singularities that can appear in Z^c. The majority of this thesis is dedicated to finding a way to emulate this construction in the case of a threefold fibred by K3 surfaces of degree two, and to studying the properties of the resulting models.

In order to do this we begin by finding the correct analogue of the relative canonical model in our case. The minimal model program suggests that the right notion to take is the *relative log canonical model* of the pair $(X, \mathcal{L})$ consisting of the threefold fibred by K3 surfaces of degree two X and its polarisation sheaf $\mathcal{L}$. This is defined as follows: first define the *relative log canonical algebra* of the pair $(X, \mathcal{L})$ to be the $\mathcal{O}_S$-algebra

$$\mathcal{R}(X, \mathcal{L}) := \bigoplus_{n \geq 0} \pi_*(\omega_X^{\otimes n} \otimes \mathcal{L}^{\otimes n}).$$

Then, if it exists, the relative log canonical model of $(X, \mathcal{L})$ is given by

$$X^c := \mathbf{Proj}_S \mathcal{R}(X, \mathcal{L}).$$

Given this, we first want to prove that the relative log canonical model exists for a threefold fibred by K3 surfaces of degree two X with polarisation $\mathcal{L}$. We show that this will follow from a version of the base point free theorem if $\mathcal{L}$ satisfies certain positivity properties. In order to use this to show existence for general pairs $(X, \mathcal{L})$, we will find a model $\pi'' \colon X'' \to S$ for X and a polarisation $\mathcal{L}''$ on X'' making $(X'', \pi'', \mathcal{L}'')$ into a threefold fibred by K3 surfaces of degree two, so that $\mathcal{L}''$ satisfies the relevant positivity properties and the relative log canonical algebras of the pairs $(X, \mathcal{L})$ and $(X'', \mathcal{L}'')$ agree. Then positivity of $\mathcal{L}''$ shows that the relative log canonical model of $(X'', \mathcal{L}'')$ exists, so we see that the relative log canonical model of the pair $(X, \mathcal{L})$ must exist also.

In order to construct $(X'', \pi'', \mathcal{L}'')$ we work locally on S and use techniques from the birational geometry of degenerations (see [FM83] and [SB83b]). However, these techniques do not necessarily produce algebraic degenerations, so in order to use them successfully we find that we first have to extend our definition of a threefold fibred by K3 surfaces of degree two to cope with cases where the threefold X may be analytic or mildly singular. Once this has been successfully achieved, we use these techniques to construct an $(X'', \pi'', \mathcal{L}'')$ with the desired properties.

With this in place we set out to find a version of Catanese's and Pignatelli's construction that works for threefolds fibred by K3 surfaces of degree two. In doing this the first thing that we note is that Catanese's and Pignatelli's construction relies heavily upon a result of Mendes-Lopes [ML89, Theorem 3.7], which classifies the canonical rings of degenerate genus two curves. In order to produce a K3 version of their construction, we first need to prove a version of this result. To do this we embark on a programme of local study of threefolds fibred by K3 surfaces of degree two.

To perform this study, we once again rely upon results from the birational geometry of degenerations to give a coarse classification of the semistable fibres that may occur in a threefold fibred by K3 surfaces of degree two. Using this coarse classification we are able to explicitly compute the images of these fibres under the map to the relative log canonical model, as complete intersections in weighted projective space. In particular, we note that these images are always contained in one of the weighted projective spaces $\mathbb{P}(1,1,1,3)$ or $\mathbb{P}(1,1,1,2,3)$. This result extends in a non-trivial way prior results of Friedman [Fri84, Theorem 2.2] and Shah [Sha80, Theorem 2.4], both of whom have previously studied the explicit form of degenerations of K3 surfaces of degree two.

Using this result we are finally able to find a version of Catanese's and Pignatelli's construction that works for threefolds fibred by K3 surfaces of degree two. We find that any threefold fibred by K3 surfaces of degree two determines a certain 5-tuple of data on the base curve S, from which its relative log canonical algebra, and hence its relative log canonical model, can be explicitly reconstructed. Furthermore, we prove an analogue of Catanese's and Pignatelli's main result [CP06, Theorem 4.13]: we show that given any 5-tuple of data satisfying certain assumptions, we can find a threefold fibred by K3 surfaces of degree two that determines that 5-tuple. From this we can see that the relative log canonical models of threefolds fibred by K3 surfaces of degree two are completely classified by their associated 5-tuples.

With this in place, the final part of this thesis calculates some of the properties of these relative log canonical models, in terms of the data coming from their associated 5-tuples. In particular we are able to calculate expressions for the canonical sheaf, Kodaira dimension and the coherent Euler characteristic. Furthermore, we give necessary and sufficient conditions for a certain resolution Y of the singularities of the relative log canonical model to be a Calabi-Yau threefold, and find an expression for the rank of its second integral cohomology $H^2(Y,\mathbb{Z})$.

We conclude this introduction with a brief overview of the contents of each of the chapters of this thesis.

In Chapter 1 we begin by giving some background information about K3 surfaces of degree two and give a formal definition of a threefold fibred by K3 surfaces of degree two. We then collect together some useful results about weighted projective bundles that will be used in the rest of the chapter. We conclude the chapter with the construction of the K3-Weierstrass model and a result (Theorem 1.3.3) proving that it exists for any threefold fibred by K3 surfaces of degree two.

Chapter 2 begins with an example illustrating the problems with the K3-Weierstrass model and discusses why the relative log canonical model is likely to be better. We then embark on a survey of relevant results from the minimal model program, culminating in Corollary 2.2.4 and Proposition 2.2.7, which give conditions for the existence of the relative log canonical model. In the next part of the chapter we give an overview of some results from the birational geometry of degenerations, before using these in the final section to prove Theorem 2.4.9, which proves the existence of the relative log canonical model for a semistable threefold fibred by K3 surfaces of degree two.

In Chapter 3 we embark upon a programme of local study of semistable threefolds fibred by K3 surfaces of degree two. We begin by discussing the theory of semi log canonical surface singularities, which will be required in order to state the main result of the chapter. We then state this result (Theorem 3.2.2), which gives an explicit description of the degenerate fibres that can occur in the relative log canonical model of a semistable threefold fibred by K3 surfaces of degree two. The remainder of the chapter is occupied by a proof of this theorem.

In Chapter 4 we are finally ready to embark upon an explicit construction for the relative log canonical algebra of a threefold fibred by K3 surfaces of degree two. We begin with a detailed analysis of the structure of this algebra, then show how this can be used to reconstruct it from a 5-tuple of data that is easily determined from the threefold fibred by K3 surfaces of degree two. Finally we prove the main result of this chapter (Theorem 4.3.2), which shows that, given any 5-tuple of data satisfying certain conditions, we may find a threefold fibred by K3 surfaces of degree two that determines that 5-tuple.

Finally, in Chapter 5 we study some of the properties of these relative log canonical models. Theorem 5.1.1, Corollary 5.1.5 and Theorem 5.2.1 give expressions for the canonical sheaf, Kodaira dimension and coherent Euler characteristic of these models respectively, in terms of the 5-tuples of data that determine them. Next, Theorem 5.3.2 gives a necessary and sufficient condition for a certain resolution Y of these models to be a Calabi-Yau threefold, again in terms of the 5-tuples of data that determine them. Finally, if the resolution Y is a Calabi-Yau threefold, Corollary 5.5.3 gives an explicit formula for the rank of its second integral cohomology group $H^2(Y, \mathbb{Z})$.

0.2 Notational Conventions

We will now outline some notational conventions that will be followed in the rest of this work:

(1) A *variety* will always refer to an algebraic variety. If we wish to work in the analytic setting, we will always refer to a *complex space* or a *complex manifold* (the latter of which will always be nonsingular, compact and connected).

(2) A *curve* (resp. *surface*) will always be a connected 1-dimensional (resp. 2-dimensional) variety (or compact complex space).

(3) Let $\mathscr{F}$ be a coherent sheaf on a variety (or complex space) X. We let $\mathscr{F}^\vee$ denote the *dual* of $\mathscr{F}$, defined by $\mathscr{F}^\vee := \mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$.

(4) Let $\mathscr{F}$ be a coherent sheaf on a variety (or compact complex space) X. Then the rank of the ith cohomology group $H^i(X, \mathscr{F})$ is finite and will be denoted by $h^i(X, \mathscr{F})$.

(5) Let $\mathcal{L}$ be a line bundle on a variety (or compact complex space) X. The notation $\mathcal{L}^n$ will always be reserved for the n-fold tensor power of $\mathcal{L}$ (normally denoted $\mathcal{L}^{\otimes n}$). If X is a surface and we wish to refer to the self-intersection number of $\mathcal{L}$, we will always write $\mathcal{L}.\mathcal{L}$.

(6) Let D and D' be two Weil divisors on a normal variety (or compact complex

space) X. If D and D' are linearly equivalent we write $D \sim D'$. A divisor D will be called *trivial* if $D \sim 0$. The group of Weil divisors modulo linear equivalence on X will be denoted $\mathrm{Cl}(X)$.

(7) A $\mathbb{Q}$-*divisor* on a normal variety (or compact complex space) X is a formal linear combination of prime Weil divisors $\sum_i d_i D_i$, with $d_i \in \mathbb{Q}$. A $\mathbb{Q}$-divisor D will be called $\mathbb{Q}$-*Cartier* if mD is Cartier for some $0 \neq m \in \mathbb{Z}$. Numerical equivalence of $\mathbb{Q}$-divisors D, D' will be denoted by $D \equiv D'$. A $\mathbb{Q}$-divisor D is called *numerically trivial* if $D \equiv 0$.

(8) A $\mathbb{Q}$-divisor D on a normal variety (or compact complex space) X is called *nef* if $D.C \geq 0$ for all irreducible curves $C \subset X$. If $\pi\colon X \to S$ is a morphism of varieties (or compact complex spaces) and D is a $\mathbb{Q}$-divisor on X, then D is called π-*nef* if $D.C \geq 0$ for all irreducible curves C that are contracted by π.

(9) A line bundle $\mathcal{L}$ on a variety (or compact complex space) X of dimension n is called *big* if

$$\limsup_{p \to \infty} p^{-n} h^0(X, \mathcal{L}^p) > 0.$$

(10) Let $\mathscr{F}$ be a coherent sheaf on a variety (or complex space) X. We call $\mathscr{F}$ *reflexive* if $\mathscr{F}^{\vee\vee} = \mathscr{F}$. A reflexive sheaf of rank one will be called *divisorial*. By the results of [Rei80, Appendix to §1], if X is a normal variety then divisorial sheaves on X are in bijective correspondence with elements of $\mathrm{Cl}(X)$. In analogy with Cartier divisors on X, we denote the divisorial sheaf corresponding to a Weil divisor D by $\mathcal{O}_X(D)$.

(11) Let D be a $\mathbb{Q}$-divisor on a normal variety (or compact complex space) X and let $f\colon X - \to Y$ be a birational map. We denote the strict transform of D under f by $f_+(D)$. Note that this can lead to the slightly unusual notation f_+^{-1}, meaning “the strict transform under f^{-1}”.

Chapter 1

Polarised K3-Fibrations and the K3-Weierstrass Model

1.1 Polarised K3 Surfaces

We begin with some general results about K3 surfaces. First, however, we need to define them:

Definition 1.1.1. *A* K3 surface *is a nonsingular surface X with trivial canonical class $\omega_X \cong \mathcal{O}_X$ and vanishing irregularity $h^1(X, \mathcal{O}_X) = 0$.*

Note that this definition does not imply that X is algebraic, but that the hypothesis $h^1(X, \mathcal{O}_X) = 0$ implies that X is Kähler [BHPvdV04, Theorem 3.1].

The following result gives some useful information about the birational invariants of K3 surfaces:

Proposition 1.1.2 [Bea96, Theorem VIII.2]. *Let X be a K3 surface. Then X has the following birational invariants:*

- *The irregularity* $q(X) = h^1(X, \mathcal{O}_X) = 0$.
- *The geometric genus* $p_g(X) = h^2(X, \mathcal{O}_X) = 1$.

Next we wish to define a *polarisation* on our K3 surfaces. As we shall see, by fixing a polarisation we may obtain a lot of information about the structure of our K3 surface.

Definition 1.1.3. *Let X be a K3 surface. A* polarisation of degree $2d$ *on X is an ample line bundle $\mathcal{L}$ on X with self-intersection number $\mathcal{L}.\mathcal{L} = 2d$. If a K3 surface admits a polarisation of degree $2d$, we call it a* K3 surface of degree $2d$.

We note that, as the polarisation $\mathcal{L}$ is assumed to be ample, any K3 surface of degree $2d$ must be projective.

In this thesis we will restrict ourselves to one of the simplest cases, where $d = 1$. The following example shows that these K3 surfaces of degree two can be described explicitly as hypersurfaces in a certain weighted projective space.

Example 1.1.4 (Hyperelliptic Case). Let X be a K3 surface and let $\mathcal{L}$ be an ample invertible sheaf with self-intersection number $\mathcal{L}.\mathcal{L} = 2$ on X. This ample invertible sheaf determines a natural isomorphism between X and $\mathrm{Proj} \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n)$ which, as we shall see, has the structure of a complete intersection in some weighted projective space (for more information on weighted projective spaces, see [IF00]). We will find out which weighted projective space, along with a general form for the equation of X, by considering the cohomology of $\mathcal{L}^n$ for $n > 0$.

By the Riemann-Roch theorem for surfaces, we have

$$\chi(\mathcal{L}^n) = \chi(\mathcal{O}_X) + \frac{1}{2}(n^2\mathcal{L}.\mathcal{L} - n\mathcal{L}.\omega_X).$$

Now, noting that $\mathcal{L}$ is ample and $\omega_X \cong \mathcal{O}_X$, for $n > 0$

$$\begin{aligned}
\chi(\mathcal{L}^n) &= h^0(X, \mathcal{L}^n) - h^1(X, \mathcal{L}^n) + h^2(X, \mathcal{L}^n) \\
&= h^0(X, \mathcal{L}^n) - h^1(X, \mathcal{L}^{-n}) + h^0(X, \mathcal{L}^{-n}) \\
&= h^0(X, \mathcal{L}^n)
\end{aligned}$$

by Serre duality and Kodaira vanishing. Similarly, by Proposition 1.1.2,

$$\chi(\mathcal{O}_X) = h^0(X, \mathcal{O}_X) - q(X) + p_g(X) = 2.$$

Finally, $\mathcal{L}.\mathcal{L} = 2$ by the polarisation information. So we have, for $n > 0$,

$$h^0(X, \mathcal{L}^n) = n^2 + 2.$$

Calculating, we get

$$\begin{aligned} h^0(X, \mathcal{L}) &= 3 \\ h^0(X, \mathcal{L}^2) &= 6 \\ h^0(X, \mathcal{L}^3) &= 11 \\ &\vdots \\ h^0(X, \mathcal{L}^6) &= 38 \end{aligned}$$

Since $h^0(X, \mathcal{L}) = 3$, the linear system of effective divisors defined by $\mathcal{L}$ is nonempty. Let H be a general member of this linear system. We can gather information about the map defined by $\mathcal{L}$ by studying the form of $|H|$.

In [May72, Proposition 1] Mayer shows that if the linear system $|H|$ has fixed points then it must have a fixed component. Furthermore as H is ample, by [May72, Proposition 8], if $|H|$ has a fixed component then the general member of $|H|$ has the form $(2E+F)$, where F is an irreducible fixed rational curve and E is an elliptic curve. These components have intersection multiplicities $E^2 = 0$, $E.F = 1$ and $F^2 = -2$. But then $H.F = 2E.F + F^2 = 0$, contradicting the Kleiman condition for the ampleness of H. Thus, $|H|$ must be base point free.

Let x_1, x_2 and x_3 be generating sections of $H^0(X, \mathcal{L})$. Since $|H|$ is base point free, the sections x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3 and x_2x_3 will generate $H^0(X,\mathcal{L}^2)$, and the ten degree 3 monomials will be linearly independent in $H^0(X,\mathcal{L}^3)$. As $H^0(X,\mathcal{L}^3)$ is 11-dimensional, we must introduce a new variable, z, with weight 3. Then, by [May72, Corollary 6], $\mathcal{L}^3$ is very ample, so $\bigoplus_{m\geq 0} H^0(X,\mathcal{L}^m)$ is generated in degree 3 and there are no new variables of higher weight. Continuing upwards, we eventually find that there are 39 degree 6 monomials in the x_i and z. But $H^0(X,\mathcal{L}^6)$ is 38-dimensional, so there must be a relation between these monomials. Thus, X is a hypersurface of degree 6 in $\mathbb{P}(1,1,1,3)$.

Conversely, let X be a well-formed quasismooth hypersurface (for definitions, see [IF00, Section 6]) of degree 6 in $\mathbb{P} := \mathbb{P}(1,1,1,3)$. By the adjunction formula

$$\begin{aligned}\mathcal{O}_X(K_X) &= \mathcal{O}_X(K_\mathbb{P}) \otimes \mathcal{O}_\mathbb{P}(X)|_X \\ &= \mathcal{O}_X(-1-1-1-3) \otimes \mathcal{O}_X(6) \\ &= \mathcal{O}_X.\end{aligned}$$

So $K_X \sim 0$. Next consider the exact sequence

$$0 \longrightarrow \mathcal{O}_\mathbb{P}(-X) \longrightarrow \mathcal{O}_\mathbb{P} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

This gives rise to the long exact sequence of cohomology

$$\cdots \longrightarrow H^1(\mathbb{P},\mathcal{O}_\mathbb{P}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^2(\mathbb{P},\mathcal{O}_\mathbb{P}(-X)) \longrightarrow \cdots$$

By [IF00, Lemma 7.1], the outer terms in the above sequence vanish, so $H^1(X,\mathcal{O}_X) = 0$. Hence, X is a K3 surface.

Now let x_1, x_2, x_3 and z be weighted co-ordinates on $\mathbb{P}(1,1,1,3)$. Completing the square in z and projecting to the space spanned by the x_i, we can realise X as a double cover of unweighted $\mathbb{P}^2$ branched over a nonsingular sextic curve. The inverse image of a hyperplane section of $\mathbb{P}^2$ under the covering map is an ample divisor on X with self-intersection number 2. Hence, X is a K3 surface of degree two.

Drawing a parallel with the classification of curves of genus two, we say that a double cover of $\mathbb{P}^2$ branched over a (possibly singular) sextic curve is *hyperelliptic*. By the argument above, any K3 surface of degree two is hyperelliptic.

Example 1.1.5 (Unigonal Case). For our second example, it will prove useful to see what happens when we relax some of the assumptions in the definition of a K3 surface of degree two. Specifically, we will study the case where the polarisation is only *pseudo-ample* (for full definitions see Dolgachev [Dol96, Section 1]).

In this case X is still a K3 surface, but the polarisation $\mathcal{L} \in \mathrm{Pic}(X)$ may not necessarily be ample. Instead we assume that $\mathcal{L}$ is nef and big. Note that this may immediately cause a problem, as without an ample invertible sheaf we have no guarantee that X will be projective. We can solve this problem by noting that $\mathcal{L}$ is big so, by [MM07, Theorem 2.2.15], X is a Moishezon manifold, and that X is also Kähler so, by [MM07, Theorem 2.2.26], X is projective.

Now, assuming that $\mathcal{L}^n$ is generated by its global sections for sufficiently large n, the invertible sheaf $\mathcal{L}$ will determine a birational morphism

$$X \longrightarrow \mathrm{Proj} \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n)$$

that contracts exactly those curves C in X with $\mathcal{L}.C = 0$. We will determine this map by calculating the cohomology of $\mathcal{L}^n$ for $n > 0$.

By a similar argument to that used in Example 1.1.4 (using the general Kodaira vanishing theorem [KM98, Theorem 2.70] for nef and big divisors), we once again obtain

$$h^0(X, \mathcal{L}^n) = n^2 + 2.$$

Since $h^0(X, \mathcal{L}) = 3$, the linear system of effective divisors defined by $\mathcal{L}$ is nonempty.

As before, we let H denote a general member of this linear system. The proof of [May72, Corollary 5] shows that $|2H|$ is base point free, so $\mathcal{L}^2$ is generated by its global sections. By studying the form of $|H|$ we can deduce information about the map defined by $\mathcal{L}$.

By [May72, Proposition 8], H can be written as either

(i) an irreducible curve with $H^2 = 2$, or

(ii) a sum $H = (2E + F)$, where E is an elliptic curve with $E^2 = 0$ and F is a fixed rational curve with $E.F = 1$ and $F^2 = -2$.

In case (i), $|H|$ is base point free. By an argument analogous to that used in Example 1.1.4, the morphism defined by $\mathcal{L}$ takes X to a (possibly singular) sextic hypersurface in $\mathbb{P}(1, 1, 1, 3)$. This morphism contracts exactly those curves C with $\mathcal{L}.C = 0$. By the Hodge index theorem, any such curve must have $C^2 < 0$ and so, since X is a nonsingular K3 surface, must be rational with $C^2 = -2$ (by adjunction). Contracting these (-2)-curves will result in at worst rational double point singularities. Hence, in this case, $\mathcal{L}$ defines a morphism from X to a sextic hypersurface in $\mathbb{P}(1, 1, 1, 3)$ that has at worst rational double point singularities, and we are still in the hyperelliptic case of Example 1.1.4.

In case (ii), as $H.F = 0$, we may contract F to give a new surface X' having an ordinary double point singularity. The image H' of H is still nef and big, but now H' is a pair of nonsingular elliptic curves meeting at the singular point of X'. This point is the unique base point of the linear system $|H'|$.

Let x_1, x_2 and x_3 be sections that generate $H^0(X', \mathcal{O}_{X'}(H'))$. By the proof of [May72, Corollary 5], $|2H'|$ is base point free, so there must be a quadric relation between the x_i in $H^0(X', \mathcal{O}_{X'}(2H'))$. Thus, we need to introduce a new variable, y, in weight 2. As in Example 1.1.4, we will need yet another new variable, z, in weight 3. By [May72, Corollary 5], the morphism defined by the linear system $|3H'|$ is birational onto its image, so $\bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n)$ is generated in degree 3 and there will be no new variables of higher weight. Continuing upwards, we find that there will be a degree 6 relation in the x_i, y and z.

Thus, $|H'|$ defines a morphism from X' to a complete intersection of bidegree $(2,6)$ in $\mathbb{P}(1,1,1,2,3)$, where the degree 2 relation does not involve the degree 2 variable. As in case (i), the only curves contracted by this morphism will be rational (-2)-curves, leading to at worst rational double point singularities in the image. So, in case (ii), $\mathcal{L}$ defines a morphism to such a complete intersection.

Note that the complete intersection of bidegree $(2,6)$ in $\mathbb{P}(1,1,1,2,3)$ as defined above cannot be seen as a double cover of $\mathbb{P}^2$, so we are no longer in the hyperelliptic case of Example 1.1.4. Instead, this complete intersection can be seen as a double cover of the quadric cone

$$\{f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2)}[x_1, x_2, x_3, y]$$

ramified over a (possibly singular) sextic curve and the vertex $(0:0:0:1)$ of the cone. Note that this sextic curve does not pass through the vertex $(0:0:0:1)$, as the elliptic curves defining the general member of $|H'|$ must be nonsingular. Drawing another parallel with the classification of curves of genus 2, we call such a double cover *unigonal*.

This example shows that any K3 surface admitting a pseudo-ample line bundle with self-intersection number two is either hyperelliptic or unigonal. Note that, in fact, both of these cases can be seen as subcases of the general complete intersection of bidegree $(2,6)$ in $\mathbb{P}(1,1,1,2,3)$. Then if the quadric relation involves the degree two variable, we can use it to eliminate this variable and reduce to the hyperelliptic case of a sextic hypersurface in $\mathbb{P}(1,1,1,3)$. The only time when such a reduction is not possible is in the unigonal case, when the quadric relation does not involve the degree two variable.

Remark 1.1.6. Here it is convenient to make a few remarks on the moduli space of polarised K3 surfaces. Whilst this will not be used in the sequel per se, it will provide us with valuable insight into the deeper theory underlying some results.

By a well-known result of Kodaira, the moduli of all algebraic K3 surfaces are parametrised by a countable union of 19-dimensional irreducible quasi-projective varieties. The components $\mathcal{S}_{2d}$ of the moduli space correspond to the K3 surfaces of degree $2d$.

Returning briefly to our example of the K3 surface of degree two, X is given by a sextic equation in $\mathbb{P}(1,1,1,3)$. This weighted projective space has 39 monomials of degree 6, so there are 38 independent complex coefficients in the equation for X. However, we may complete the square in the weight 3 variable to remove 10 of these. Furthermore, $PGL(3,\mathbb{C})$ acts upon the three weight 1 variables to give 8 further dependence relations between the monomials. This leaves 20 independent complex coefficients in the defining equation for X, which define a 19-dimensional projective space. This agrees nicely with our observation above that the moduli of K3 surfaces of degree two is 19-dimensional.

Finally, as this work will be mainly concerned with constructing explicit models for threefolds fibred by polarised K3 surfaces, we conclude this section by extending the definition of a polarised K3 surface to a threefold that admits a fibration by K3 surfaces.

Definition 1.1.7. *Let S be a nonsingular complex curve. A* threefold fibred by K3 surfaces of degree $2d$ *over S, denoted $(X,\pi,\mathcal{L})$, consists of:*

(1) *A three dimensional nonsingular complex variety X;*

(2) *A projective, flat, surjective morphism $\pi\colon X \to S$ with connected fibres, whose general fibres are K3 surfaces;*

(3) *An invertible sheaf $\mathcal{L}$ on X that induces an ample invertible sheaf $\mathcal{L}_s$ with self-intersection number $\mathcal{L}_s.\mathcal{L}_s = 2d$ on a general fibre X_s of $\pi\colon X \to S$.*

Remark 1.1.8. In the sequel (Section 2.4) we will extend this definition to encompass the cases where X is a normal complex variety with mild singularities or a nonsingular compact complex manifold. However, before we can do this we will need to develop some more theoretical tools. We refer the interested reader to Definition 2.4.2 and Definition 2.4.5.

1.2 Weighted Projective Bundles

In Example 1.1.4, we saw that a general K3 surface of degree two can be embedded into the weighted projective space $\mathbb{P}(1,1,1,3)$. If we wish to construct a fibration by K3 surfaces of degree two, it seems sensible to start with a fibration by weighted projective spaces and then embed our K3 surfaces into the fibres. This leads us naturally to the concept of a *weighted projective bundle*. In this section, we aim to construct weighted projective bundles using a mild generalisation of the method used by Mullet [Mul09, Section 4], then give a few of their properties that will be used in what follows.

We begin with the definitions:

Definition 1.2.1 [Mul09, 4.1]. *Let S be a scheme, and let $(a_0, \ldots, a_n)$ be a sequence of strictly positive integers. Define a* weighted locally free sheaf *with weights $(a_0, \ldots, a_n)$ to be a locally free sheaf of $\mathcal{O}_S$-modules $\mathcal{E}$ together with an ordered decomposition of $\mathcal{E}$ as $\mathcal{E} \cong \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_n$, where each $\mathcal{E}_i$ is a locally free sheaf and the direct sum is to be interpreted as a graded sheaf with $\mathcal{E}_i$ placed in degree a_i for $0 \leq i \leq n$.*

This enables us to define:

Definition 1.2.2 [Mul09, 4.3]. *Let S be a scheme. Given a weighted locally free sheaf $\mathcal{E}$ with weights $(a_0, \ldots, a_n)$, let $\widetilde{\mathrm{Sym}}(\mathcal{E})$ denote the weighted symmetric algebra of $\mathcal{E}$, where we insist that $\mathcal{E}_i$ have homogeneous degree a_i in $\widetilde{\mathrm{Sym}}(\mathcal{E})$. We define the* weighted projective bundle *associated to $\mathcal{E}$ to be the S-scheme*

$$\tilde{\mathbb{P}}_S(\mathcal{E}) := \mathbf{Proj}_S(\widetilde{\mathrm{Sym}}(\mathcal{E})) \xrightarrow{p} S.$$

The first thing that we wish to show about weighted projective bundles is that the fibres are indeed the weighted projective spaces that we want. We have:

Lemma 1.2.3 [Mul09, Lemma 4.4]. *Let S be a nonsingular variety over $\mathbb{C}$, and let $\mathcal{E} \cong \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_n$ be a weighted locally free sheaf on S with weights $(a_0, \ldots, a_n)$. Then the weighted projective bundle $\tilde{\mathbb{P}}_S(\mathcal{E})$ is a locally trivial fibre bundle over S with fibre the weighted projective space $\mathbb{P}(a_0, \ldots, a_0, a_1, \ldots, a_{n-1}, a_n, \ldots, a_n)$, where a_i appears with multiplicity* $\mathrm{rank}(\mathcal{E}_i)$.

Next, we would like to be able to place a condition on our weighted projective bundles to ensure that they are well-behaved. This is the analogue of *well-formedness* [IF00, 5.11] for a weighted projective space:

Definition 1.2.4. *A weighted projective bundle $\tilde{\mathbb{P}}_S(\mathcal{E})$ is* well-formed *if S is nonsingular and for all choices of j*

$$\mathrm{hcf}(a_0, \ldots, a_0, a_1, \ldots, a_j, \widehat{a_j}, a_j, \ldots, a_{n-1}, a_n, \ldots, a_n) = 1,$$

where each a_i appears with multiplicity $\mathrm{rank}(\mathcal{E}_i)$ *and one of the a_j is skipped.*

Note that, along with Lemma 1.2.3, this implies immediately that a well-formed weighted projective bundle is normal and its fibres are well-formed weighted projective spaces.

Now assume that $\tilde{\mathbb{P}}_S(\mathcal{E})$ is a well-formed weighted projective bundle over a nonsingular complex variety S. Then over any affine open set $U \subset S$, the Proj construction gives a divisorial sheaf $\mathcal{O}(1)$ on $p^{-1}(U) = \mathrm{Proj}(\widetilde{\mathrm{Sym}}(\mathcal{E})(U))$. These divisorial sheaves glue to give a divisorial sheaf $\mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(1)$ on $\tilde{\mathbb{P}}_S(\mathcal{E})$.

For all $d \geq 0$, define divisorial sheaves

$$\mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(d) := \left(\mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(1)^{\otimes d}\right)^{\vee\vee}.$$

Then for any open affine $U \subset S$, the restriction of $\mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(d)$ to $p^{-1}(U)$ is exactly $\mathcal{O}(d)$. Using this we see that if $l = \mathrm{lcm}(a_0, \ldots, a_n)$, then $\mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(l)$ is an invertible sheaf on $\tilde{\mathbb{P}}_S(\mathcal{E})$.

With this in place, we return our attention to the structure of the weighted projective bundle. We have the following analogue of [Har77, Proposition 7.10]:

Lemma 1.2.5. *Let $\tilde{\mathbb{P}}_S(\mathcal{E})$ be a well-formed weighted projective bundle over a nonsingular complex variety S, with projection map p. Then p is a proper morphism. Furthermore, if we suppose that S admits an ample invertible sheaf, then p must be projective.*

Proof. (Based on the proof of [Har77, Proposition 7.10]) We begin by showing that p is proper. For each affine open set $U \subset S$, the restriction $p|_U \colon \mathrm{Proj}(\widetilde{\mathrm{Sym}}(\mathcal{E})(U)) \to U$ is projective, so proper. But the condition of properness is local on the base, so p must be proper.

Next we show that if S admits an ample invertible sheaf, then p must be projective. Define a new graded algebra $\widetilde{\mathrm{Sym}}(\mathcal{E})^{(m)}$ by

$$\widetilde{\mathrm{Sym}}(\mathcal{E})^{(m)} := \bigoplus_{i \geq 0} \widetilde{\mathrm{Sym}}(\mathcal{E})_{im},$$

where $\widetilde{\mathrm{Sym}}(\mathcal{E})_d$ denotes the graded piece of degree d. Then the inclusion of $\widetilde{\mathrm{Sym}}(\mathcal{E})^{(m)}$ into $\widetilde{\mathrm{Sym}}(\mathcal{E})$ induces an isomorphism

$$\mathbf{Proj}_S \widetilde{\mathrm{Sym}}(\mathcal{E}) \cong \mathbf{Proj}_S\big(\widetilde{\mathrm{Sym}}(\mathcal{E})^{(m)}\big),$$

called the *m-uple embedding.*

Furthermore, if m is divisible by $\mathrm{lcm}(a_0, \ldots, a_n)$ then $\widetilde{\mathrm{Sym}}(\mathcal{E})^{(m)}$ is generated in degree one, so we may apply [Har77, Proposition 7.10] to get that p is projective. $\square$

Next we would like to know more about the sheaves $\mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(d)$ for $d \in \mathbb{N}_0$. We have the following analogue of [Har77, Proposition II.7.11]:

Lemma 1.2.6. *Let $\tilde{\mathbb{P}}_S(\mathcal{E})$ be a well-formed weighted projective bundle over a nonsingular complex variety S. Then*

(a) *If $\operatorname{rank}(\mathcal{E}) \geq 2$, there is a canonical isomorphism of graded $\mathcal{O}_S$-algebras*

$$\widetilde{\operatorname{Sym}}(\mathcal{E}) \cong \bigoplus_{d \geq 0} p_*\big(\mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(d)\big)$$

where the grading on the right hand side is given by d.

(b) *There exists a natural morphism of graded $\mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}$-algebras*

$$p^*\big(\widetilde{\operatorname{Sym}}(\mathcal{E})\big) \longrightarrow \bigoplus_{d \geq 0} \mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(d),$$

which is surjective in degree d if d is divisible by $\operatorname{lcm}(a_0, \ldots, a_n)$.

Proof. Part (a) is just a rephrasing of [Har77, Proposition II.7.11(a)] in the weighted setting.

For part (b), taking the inverse image by p of the isomorphism in (a) we get an isomorphism of graded $\mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}$-algebras

$$p^*\big(\widetilde{\operatorname{Sym}}(\mathcal{E})\big) \cong p^*p_*\Big(\bigoplus_{d \geq 0} \mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(d)\Big).$$

Composing with the natural morphism

$$p^*p_*\Big(\bigoplus_{d \geq 0} \mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(d)\Big) \longrightarrow \bigoplus_{d \geq 0} \mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(d)$$

gives the required morphism.

Finally, the fact that this morphism is surjective in degree d when d is a multiple of $\operatorname{lcm}(a_0, \ldots, a_n)$ is a relative version of the fact that the invertible sheaf $\mathcal{O}(d)$ on the weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ is generated by its global sections for d divisible by $\mathrm{lcm}(a_0, \dots, a_n)$. □

Using this, in a manner analogous to [Har77, Proposition 7.12] we can find a condition for a scheme to admit a morphism to $\tilde{\mathbb{P}}_S(\mathcal{E})$:

Proposition 1.2.7. *Let $\tilde{\mathbb{P}}_S(\mathcal{E})$ be a well-formed weighted projective bundle over a non-singular complex variety S. Suppose that $\pi\colon X \to S$ is a morphism. Let $\mathcal{L}$ be an invertible sheaf on X. Suppose that we have a morphism of graded $\mathcal{O}_X$-algebras*

$$\pi^*(\widetilde{\mathrm{Sym}}(\mathcal{E})) \longrightarrow \bigoplus_{d \geq 0} \mathcal{L}^d$$

that is surjective in degree d when d is divisible by $l = \mathrm{lcm}(a_0, \dots, a_n)$. Then there exists a morphism $\mu\colon X \to \tilde{\mathbb{P}}_S(\mathcal{E})$ over S satisfying $\mu^*(\mathcal{O}_{\tilde{\mathbb{P}}_S(\mathcal{E})}(l)) \cong \mathcal{L}^l$.

Proof. Using the notation in the proof of Lemma 1.2.5, we have an isomorphism

$$\mathbf{Proj}_S \, \widetilde{\mathrm{Sym}}(\mathcal{E}) \cong \mathbf{Proj}_S \big(\widetilde{\mathrm{Sym}}(\mathcal{E})^{(l)}\big),$$

given by the l-uple embedding. Furthermore, $\widetilde{\mathrm{Sym}}(\mathcal{E})^{(l)}$ is generated in degree one and we have a surjective morphism of graded algebras

$$\pi^* \widetilde{\mathrm{Sym}}(\mathcal{E})^{(l)} \longrightarrow \bigoplus_{d \geq 0} \mathcal{L}^{dl}.$$

Then an argument similar to that used in the proof of [Har77, Proposition 7.12] gives a morphism

$$\tilde{\mu}\colon X \longrightarrow \mathbf{Proj}_S \big(\widetilde{\mathrm{Sym}}(\mathcal{E})^{(l)}\big)$$

satisfying $\tilde{\mu}^*(\mathcal{O}(1)) \cong \mathcal{L}^l$. Composing with the l-uple embedding gives the result. □

Finally, we conclude this section with a weighted analogue of [Har77, Proposition III.9.8]. This will enable us to extend a flat family of hypersurfaces in a weighted projective bundle over a punctured curve to a flat family over the whole curve; we will need to do this in the construction of our K3-fibrations.

Lemma 1.2.8. *Let S be a nonsingular variety of dimension 1 over $\mathbb{C}$, and let $P \in S$ be a closed point. Let $\mathcal{E}$ be a weighted locally free sheaf on S, and let $X \subset \tilde{\mathbb{P}}_{S \setminus P}(\mathcal{E})$ be a closed subscheme which is flat over $S \setminus P$. Then there exists a unique closed subscheme $\overline{X} \subset \tilde{\mathbb{P}}_S(\mathcal{E})$, flat over S, whose restriction to $\tilde{\mathbb{P}}_{S \setminus P}(\mathcal{E})$ is X.*

Proof. (Following the proof of [Har77, Proposition III.9.8]). Let $\overline{X}$ be the scheme-theoretic closure of X in $\tilde{\mathbb{P}}_S(\mathcal{E})$. By [Har77, Proposition III.9.7], the natural morphism $\pi \colon \overline{X} \to S$ is flat if and only if every associated point $x \in \overline{X}$ maps to the generic point of S. But the associated points of $\overline{X}$ are just those of X, so since X is flat over $S \setminus P$, $\overline{X}$ is flat over S. Furthermore, $\overline{X}$ is unique, as any other extension of X to $\tilde{\mathbb{P}}_S(\mathcal{E})$ would have some associated points mapping to P. $\square$

1.3 The K3-Weierstrass Model

We are now ready to begin constructing our first model for K3-fibrations. In order to do this we look to the well-developed theory of elliptic fibrations for inspiration. Using the fact that any elliptic curve can be embedded by an equation of Weierstrass form in $\mathbb{P}^2$, Nakayama [Nak88] proves that any complex variety that admits an elliptic fibration with a section has a birational morphism to a projective model, called the *Weierstrass model*. This model is constructed by defining a $\mathbb{P}^2$-bundle over the base variety, then taking a hypersurface in it defined by an equation of Weierstrass form. We attempt to perform an analogous construction for threefolds fibred by K3 surfaces of degree two, by embedding a flat family of K3 surfaces of degree two into a $\mathbb{P}(1,1,1,3)$-bundle. We begin by constructing the weighted projective bundle into which we can embed our fibration.

Let S be a connected nonsingular 1-dimensional variety over $\mathbb{C}$. Let $\mathcal{E}_1$ be a rank

3 vector bundle and $\mathcal{E}_3^+$ a line bundle on S (we make this unusual choice of notation for consistency with Chapter 4, it will make more sense there!). We treat $\mathcal{E}_1 \oplus \mathcal{E}_3^+$ as a weighted locally free sheaf with weights $(1, 3)$ (in the sense of Definition 1.2.2) and define $Y := \tilde{\mathbb{P}}_S(\mathcal{E}_1 \oplus \mathcal{E}_3^+)$. Denote by $p\colon Y \to S$ the natural projection and $\mathcal{O}_Y(1)$ the tautological divisorial sheaf on Y.

By Lemma 1.2.3, Y is a locally trivial fibre bundle over S with fibre the weighted projective space $\mathbb{P}(1, 1, 1, 3)$. Our next task is to construct a divisor W on Y whose intersection with a general fibre is a hypersurface of degree 6; then according to Example 1.1.4 these hypersurfaces will be K3 surfaces of degree two, as required.

Unfortunately this will turn out not to be as simple as we would like. The problem arises because we only know that the polarisation sheaf on a threefold fibred by K3 surfaces of degree two is ample on a general fibre. In practice, this means that there may exist isolated fibres that are not double covers of $\mathbb{P}^2$: for instance, the unigonal case of Example 1.1.5 may occur. For this reason, we can only explicitly construct W on a dense open subset of Y. To define W everywhere, we are forced to use the flatness property to extend W from this dense open set to all of Y.

Consider $\mathbb{P}_S(\mathrm{Sym}^6(\mathcal{E}_1) \oplus \mathcal{O}_S)$. There is an open embedding

$$\mathrm{Sym}^6(\mathcal{E}_1) \hookrightarrow \mathbb{P}_S\big(\mathrm{Sym}^6(\mathcal{E}_1) \oplus \mathcal{O}_S\big)$$

given by $v \mapsto [v, 1]$; we henceforth identify $\mathrm{Sym}^6(\mathcal{E}_1)$ with its image under this embedding. Let

$$a\colon (\mathcal{E}_3^+)^2 \longrightarrow \mathbb{P}_S\big(\mathrm{Sym}^6(\mathcal{E}_1) \oplus \mathcal{O}_S\big)$$

be a sheaf homomorphism such that $\mathrm{im}(a) \cap \mathrm{Sym}^6(\mathcal{E}_1) \neq \emptyset$. Then there exists an open set $S_0 \subset S$ such that $\mathrm{im}(a) \subset \mathrm{Sym}^6(\mathcal{E}_1)$ on S_0. Hence, restricting to S_0, we have $a \in \Gamma(S_0, (\mathcal{E}_3^+)^{-2} \otimes \mathrm{Sym}^6(\mathcal{E}_1))$. Let $p_0\colon Y_0 \to S_0$ denote the restriction of p to S_0. Let $a' \in \Gamma(Y_0, p_0^*(\mathcal{E}_3^+)^{-2} \otimes \mathrm{Sym}^6(p_0^*\mathcal{E}_1))$ denote the inverse image of a under p_0.

Let x and z be the sections of $p^*\mathcal{E}_1^\vee \otimes \mathcal{O}_Y(1)$ and $p^*(\mathcal{E}_3^+)^{-1} \otimes \mathcal{O}_Y(3)$ corresponding to the natural morphisms

$$p^*\mathcal{E}_1 \longrightarrow \mathcal{O}_Y(1)$$
$$p^*\mathcal{E}_3^+ \longrightarrow \mathcal{O}_Y(3)$$

given by Lemma 1.2.6. Denote by $W_0^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a)$ the divisor on Y_0 defined by the equation $z^2 - a'x^6$. Then $W_0 := W_0^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a)$ is flat over S_0 so, by Lemma 1.2.8, there exists a unique closed subscheme $W^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a) \subset Y$ whose restriction to Y_0 is W_0.

Definition 1.3.1. $W^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a)$ *is called the* degree two K3-Weierstrass model *of type* $(\mathcal{E}_1, \mathcal{E}_3^+, a)$ *over* S.

$W := W^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a)$ has the following properties:

(1) W is a normal complex variety and $p\colon W \to S$ is a projective, flat, surjective morphism whose general fibres are irreducible sextic hypersurfaces in $\mathbb{P}(1,1,1,3)$.

(2) The restriction $\mathcal{O}_{W_s}(1)$ of the divisorial sheaf $\mathcal{O}_W(1)$ to a general fibre W_s is an ample invertible sheaf with self-intersection number 2.

(3) The morphisms $p_0^*\mathcal{E}_1 \to \mathcal{O}_{W_0}(1)$ and $p_0^*\mathcal{E}_3^+ \to \mathcal{O}_{W_0}(3)$ obtained from Lemma 1.2.6 give sections

$$f \in \Gamma(S_0,\, \mathcal{E}_1^\vee \otimes p_*\mathcal{O}_{W_0}(1))$$
$$g \in \Gamma(S_0,\, (\mathcal{E}_3^+)^{-1} \otimes p_*\mathcal{O}_{W_0}(3)).$$

(4) Using Lemma 1.2.6, we can identify on S_0

$$p_*\mathcal{O}_{W_0}(1) = (\mathcal{E}_1|_{S_0}).f$$

$$\begin{aligned}
p_*\mathcal{O}_{W_0}(2) &= \mathrm{Sym}^2(\mathcal{E}_1|_{S_0}).f^2 \\
p_*\mathcal{O}_{W_0}(3) &= \mathrm{Sym}^3(\mathcal{E}_1|_{S_0}).f^3 \oplus (\mathcal{E}_3^+|_{S_0}).g \\
&\ \ \vdots \\
p_*\mathcal{O}_{W_0}(6) &= \mathrm{Sym}^6(\mathcal{E}_1|_{S_0}).f^6 \oplus \big(\mathrm{Sym}^3(\mathcal{E}_1|_{S_0}) \otimes \mathcal{E}_3^+|_{S_0}\big).f^3 g
\end{aligned}$$

with the relation

$$g^2 = af^6 \quad \text{in} \quad \Gamma(S_0,\, (\mathcal{E}_3^+)^{-2} \otimes p_*\mathcal{O}_{W_0}(6)).$$

Remark 1.3.2. Note that the bundle $(\mathcal{E}_3^+)^{-2} \otimes \mathrm{Sym}^6(\mathcal{E}_1)$ of which a is a section has rank 28. This seems incongruent with the observation in Section 1.1 that the moduli space of K3 surfaces of degree two is a 19-dimensional projective variety. However, we note that $\mathrm{PGL}(3,\mathbb{C})$ acts upon $\tilde{\mathbb{P}}_S(\mathcal{E}_1 \oplus \mathcal{E}_3^+)$, so that 8 of these dimensions correspond to linear transformations of the ambient space. Taking this into account, along with the 1-dimension difference caused by the change from affine to projective space, the dimensions agree perfectly.

With our construction complete, we would like to prove an analogue of Nakayama's theorem on the generality of Weierstrass models [Nak88, Theorem 2.1]; this states that any elliptic fibration that admits a section has a proper birational morphism to a Weierstrass model. Unfortunately, the flat extension contained in our construction means that we cannot guarantee quite as much for our fibrations; rather, we must forgo the proper morphism of Nakayama's result in favour of a more general birational map. We have:

Theorem 1.3.3. *Let S be a nonsingular curve. Let $(X, \pi, \mathcal{L})$ be a threefold fibred by K3 surfaces of degree two over S. Then there exists a degree two K3-Weierstrass model $W := W^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a)$ over S and a birational map $\mu \colon X - \to W$ commuting with the projections to S.*

Furthermore, there exists a dense open subset $S_0 \subset S$ such that the restriction μ_0 of μ to $X_0 := \pi^{-1}(S_0)$ is an isomorphism and $\mu_0^*\mathcal{O}_{W_0}(1) = \mathcal{L}|_{X_0}$.

Proof. We begin by gathering some information about the direct image sheaves $\pi_*\mathcal{L}^n$. These will then be used to define the vector bundles $\mathcal{E}_1$ and $\mathcal{E}_3^+$ needed to construct our K3-Weierstrass model. We prove the first result in considerably more generality than we need, as we will use it again later.

Lemma 1.3.4. *Let $\pi \colon X \to S$ be a proper surjective morphism from a normal complex variety (or complex manifold) X to a nonsingular curve S. Then if $\mathcal{L}$ is a line bundle on X, the direct image $\pi_*\mathcal{L}$ is a locally free sheaf on S.*

Proof. Firstly note that since π is a proper morphism, by a well-known theorem of Grothendieck (or Grauert in the analytic case), $\pi_*\mathcal{L}$ is coherent for all $n \geq 0$. Then since S is a nonsingular variety of dimension 1, by [Har80, Corollary 1.4 and Proposition 1.6], $\pi_*\mathcal{L}$ is locally free if and only if it is torsion-free.

Now, by definition, the direct image $\pi_*\mathcal{L}$ is torsion-free if and only if the restriction maps $H^0(U, \pi_*\mathcal{L}) \to H^0(U', \pi_*\mathcal{L})$ are injective for all pairs of open subsets U, U' that satisfy $\emptyset \neq U' \subset U \subset S$. Let U, U' be two such open sets. Then, by definition of the direct image, $H^0(U, \pi_*\mathcal{L}) \cong H^0(\pi^{-1}(U), \mathcal{L})$, and a similar statement holds for U'. Furthermore, as π is surjective, $\emptyset \neq \pi^{-1}(U') \subset \pi^{-1}(U) \subset X$. Thus, the torsion-freeness of $\pi_*\mathcal{L}$ follows immediately from the torsion-freeness of $\mathcal{L}$. □

Lemma 1.3.5. *With assumptions as in Theorem 1.3.3, $\pi_*\mathcal{L}^n$ is a locally free sheaf on S of rank $r(n)$ for $n \geq 1$, where*

$$r(n) = n^2 + 2.$$

Proof. The fact that $\pi_*\mathcal{L}^n$ is locally free follows from Lemma 1.3.4. It only remains to find the ranks of these locally free sheaves. Let s be a closed point in S, with fibre X_s

over s a K3 surface of degree two (such points form a dense open set in S). Then, by [Har77, Exercise II.5.8]

$$r(n) = \dim_{k(s)} \pi_*(\mathcal{L}^n)_s \otimes_{\mathcal{O}_s} k(s)$$

where $k(s) = \mathcal{O}_s/\mathfrak{m}_s$ is the residue field at the point s. We will use the theorem on cohomology and base change [Mum70, Corollary II.5.2] to calculate this dimension.

Let $\mathcal{L}_{X_s}$ denote the invertible sheaf induced on X_s by $\mathcal{L}$. We note that, as X_s is a K3-surface, the canonical sheaf $\omega_{X_s} \cong \mathcal{O}_X$. So the cohomology groups

$$H^1(X_s, \mathcal{L}^n_{X_s}) \cong H^1(X_s, \omega_{X_s} \otimes \mathcal{L}^n_{X_s})$$

are isomorphic. The second of these groups vanishes for $n \geq 1$ by Kodaira vanishing, as $\mathcal{L}_{X_s}$ is ample.

As a general fibre of $\pi\colon X \to S$ is a K3 surface of degree two, this vanishing holds for all s' in some neighbourhood of s. Furthermore, $\mathcal{L}^n$ is flat over S_0 so, by the theorem on cohomology and base change, the map

$$(\pi_*\mathcal{L}^n)_s \otimes_{\mathcal{O}_s} k(s) \longrightarrow H^0(X_s, \mathcal{L}^n_{X_s})$$

is an isomorphism. Thus, we have

$$\dim_{k(s)} \pi_*(\mathcal{L}^n)_s \otimes_{\mathcal{O}_s} k(s) = h^0(X_s, \mathcal{L}^n_{X_s}).$$

But the rank of this cohomology group was calculated in Example 1.1.4. Hence we have $r(n) = n^2 + 2$. □

Our next task is to define the vector bundles $\mathcal{E}_1$ and $\mathcal{E}_3^+$ needed to construct the

K3-Weierstrass model. $\mathcal{E}_1$ is easy to define,

$$\mathcal{E}_1 := \pi_*\mathcal{L}.$$

By Lemma 1.3.5, $\mathcal{E}_1$ is a locally free sheaf on S of rank 3, as required by the construction. Unfortunately, $\mathcal{E}_3^+$ is not quite so simple to define!

Let $f \in \Gamma(S, \mathcal{E}_1^\vee \otimes \pi_*\mathcal{L})$ denote the section corresponding to the identity homomorphism on $\pi_*\mathcal{L}$. Then f^3 induces a map

$$f^3 \colon \mathrm{Sym}^3(\mathcal{E}_1) \longrightarrow \pi_*\mathcal{L}^3.$$

Note that f^3 cannot be surjective; by Lemma 1.3.5, $\mathrm{Sym}^3(\mathcal{E}_1)$ has rank 10, whereas $\pi_*\mathcal{L}^3$ has rank 11. We would like to set $\mathcal{E}_3^+$ to be the cokernel of this map. However, we cannot guarantee that the sheaf obtained this way will be locally free. Hence, we are forced to adopt a more complicated definition.

Let $\mathcal{Q} := \ker(f^3)$ and $\mathcal{T}_3 := \mathrm{coker}(f^3)$. Define $\mathcal{E}_3^+$ as the reflexivisation of $\mathcal{T}_3$,

$$\mathcal{E}_3^+ := (\mathcal{T}_3)^{\vee\vee}.$$

Then since $\mathcal{E}_3^+$ has rank one and S is nonsingular, by a result of Hartshorne [Har80, Proposition 1.9], $\mathcal{E}_3^+$ is locally free.

These sheaves all fit together in a commutative diagram with exact rows and columns on S, as shown in Figure 1.1.

We would ideally like to express $\pi_*\mathcal{L}^3$ as a direct sum involving $\mathrm{Sym}^3\mathcal{E}_1$ and $\mathcal{E}_3^+$. Unfortunately, the sheaves $\mathcal{Q}$, $\mathrm{Tors}(\mathcal{T}_3)$ and $\mathcal{R}$ prevent us from doing this. However, we can partly recover the situation by showing that these sheaves vanish on an open set of S.

Lemma 1.3.6. *There exists a dense open set $S_0 \subset S$ such that $\mathcal{Q}|_{S_0} = 0$ and $\mathcal{T}_3|_{S_0}$ is*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & & \\
 & & \mathcal{Q} \arrow[d] & 0 \arrow[d] & \mathrm{Tors}(\mathcal{T}_3) \arrow[d] & & \\
0 \arrow[r] & \mathcal{Q} \arrow[r] & \mathrm{Sym}^3(\mathcal{E}_1) \arrow[r, "f^3"] \arrow[d] & \pi_*\mathcal{L}^3 \arrow[r] \arrow[d, equal] & \mathcal{T}_3 \arrow[r] \arrow[d] & 0 & \\
 & 0 \arrow[r] & \ker(q) \arrow[r] \arrow[d] & \pi_*\mathcal{L}^3 \arrow[r, "q"] \arrow[d] & \mathcal{E}_3^+ \arrow[r] \arrow[d] & \mathcal{R} \arrow[r] & 0 \\
 & & \mathrm{Tors}(\mathcal{T}_3) \arrow[d] & 0 & \mathcal{R} \arrow[d] & & \\
 & & 0 & & 0 & &
\end{tikzcd}$$

Figure 1.1.

locally free of rank one.

Proof. Let S_0 be the dense open set of points over which the fibres of π are K3 surfaces of degree two.

From the commutative diagram above, we have an exact sequence of sheaves on S

$$\mathrm{Sym}^3(\mathcal{E}_1) \xrightarrow{f^3} \pi_*\mathcal{L}^3 \longrightarrow \mathcal{T}_3 \longrightarrow 0$$

Let $s \in S_0$ be a closed point. Localising to stalks and tensoring with $k(s)$, we have

$$\mathrm{Sym}^3(\mathcal{E}_1)_s \otimes_{\mathcal{O}_s} k(s) \xrightarrow{f_s^3} (\pi_*\mathcal{L}^3)_s \otimes_{\mathcal{O}_s} k(s) \longrightarrow (\mathcal{T}_3)_s \otimes_{\mathcal{O}_s} k(s) \longrightarrow 0$$

By the argument detailed in Lemma 1.3.5, using the theorem on cohomology and base change, we have for $n \geq 1$

$$(\pi_*\mathcal{L}^n)_s \otimes_{\mathcal{O}_s} k(s) \cong H^0(X_s, \mathcal{L}_s^n).$$

Then, since X_s is a K3 surface of degree two with polarisation $\mathcal{L}_s \in \mathrm{Pic}(X_s)$, by Example 1.1.4 we have a natural injection

$$\mathrm{Sym}^3 H^0(X_s, \mathcal{L}_s) \hookrightarrow H^0(X_s, \mathcal{L}_s^3)$$

which induces f_s^3 under the above isomorphism. So for any $s \in S_0$, the sequence

$$0 \longrightarrow \mathrm{Sym}^3(\mathcal{E}_1)_s \otimes_{\mathcal{O}_s} k(s) \xrightarrow{f_s^3} (\pi_*\mathcal{L}^3)_s \otimes_{\mathcal{O}_s} k(s) \longrightarrow (\mathcal{T}_3)_s \otimes_{\mathcal{O}_s} k(s) \longrightarrow 0$$

is exact. Since $\mathcal{E}_1$ and $\pi_*\mathcal{L}^3$ are locally free, the dimensions (over $k(s)$) of the first two terms in this sequence are constant across all $s \in S_0$ [Har77, Exercise II.5.8]. So, by exactness, the dimension of the last term in the sequence is constant also. Hence, by [Har77, Exercise II.5.8] again, $\mathcal{T}_3$ is locally free on S_0. Furthermore, since f_s^3 is injective on S_0, the kernel $\mathcal{Q}|_{S_0} = 0$. Finally, by the calculation in Lemma 1.3.5,

$$\dim_{k(s)}(\mathcal{T}_3)_s \otimes_{\mathcal{O}_s} k(s) = 1$$

on S_0. So $(\mathcal{T}_3)|_{S_0}$ has rank one. □

Let $S_0 \subset S$ be as in the lemma and $X_0 \subset X$ be its inverse image under π. Since $(\mathcal{T}_3)|_{S_0}$ is locally free, $\mathcal{T}_3 \cong (\mathcal{T}_3)^{\vee\vee}$ on S_0, so $\mathrm{Tors}(\mathcal{T}_3)$ and $\mathcal{R}$ are supported on the complement of S_0 in S.

Hence, we have the following exact sequence on S_0:

$$0 \longrightarrow \mathrm{Sym}^3(\mathcal{E}_1) \xrightarrow{f^3} \pi_*\mathcal{L}^3 \xrightarrow{q} \mathcal{E}_3^+ \longrightarrow 0$$

Our next task is to show that this sequence splits on S_0. This will finally give us the direct sum decomposition of $\pi_*\mathcal{L}^3$ that we desire.

Proposition 1.3.7. *Let $\varphi\colon \pi_*\mathcal{L}^3 \otimes (\mathcal{E}_3^+)^{-1} \to \mathcal{O}_{S_0}$ be the homomorphism on S_0 obtained* *by tensoring $(\mathcal{E}_3^+)^{-1}$ with $q\colon \pi_*\mathcal{L}^3 \to \mathcal{E}_3^+$. There is a section $g \in \Gamma(S_0, (\mathcal{E}_3^+)^{-1} \otimes \pi_*\mathcal{L}^3)$ such that:*

(i) $\varphi(g) = 1$;

(ii) *There is an expression $g^2 = af^6$ in $\Gamma(S_0, (\mathcal{E}_3^+)^{-2} \otimes \pi_*\mathcal{L}^6)$ for some section $a \in \Gamma(S_0, (\mathcal{E}_3^+)^{-2} \otimes \mathrm{Sym}^6(\mathcal{E}_1))$.*

Proof. The proof of this proposition is based upon the argument given by Nakayama in the proof of [Nak88, Proposition 2.3]. We begin by constructing the section g locally on S_0, then extend to the whole of S_0 by showing uniqueness of the local sections.

Let U be an open subset of S_0 such that $\mathcal{E}_1|_U$ and $\mathcal{E}_3^+|_U$ are trivial. Then we can take a section $g \in \Gamma(U, (\mathcal{E}_3^+)^{-1} \otimes \pi_*\mathcal{L}^3)$ such that $\varphi(g) = 1$. We have the following equalities:

$$\begin{aligned}
(\pi_*\mathcal{L})|_U &= (\mathcal{E}_1|_U).f \\
(\pi_*\mathcal{L}^2)|_U &= \mathrm{Sym}^2(\mathcal{E}_1|_U).f^2 \\
(\pi_*\mathcal{L}^3)|_U &= \mathrm{Sym}^3(\mathcal{E}_1|_U).f^3 \oplus (\mathcal{E}_3^+|_U).g \\
&\vdots \\
(\pi_*\mathcal{L}^6)|_U &= \mathrm{Sym}^6(\mathcal{E}_1|_U).f^6 \oplus \big(\mathrm{Sym}^3(\mathcal{E}_1|_U) \otimes \mathcal{E}_3^+|_U\big).f^3 g.
\end{aligned}$$

So there exist $a \in \Gamma(U, (\mathcal{E}_3^+)^{-2} \otimes \mathrm{Sym}^6(\mathcal{E}_1))$ and $b \in \Gamma(U, (\mathcal{E}_3^+)^{-1} \otimes \mathrm{Sym}^3(\mathcal{E}_1))$ such that $g^2 = af^6 + bf^3 g$. Put

$$g' := g - \frac{1}{2} b f^3.$$

Then $g'^2 = af^6$ for some $a \in \Gamma(U, (\mathcal{E}_3^+)^{-2} \otimes \mathrm{Sym}^6(\mathcal{E}_1))$. Therefore, a g exists on U that satisfies the proposition. We will now show that such a g is unique, which will allow us to patch to give a single global section.

Suppose $G \in \Gamma(U, (\mathcal{E}_3^+)^{-1} \otimes \pi_*\mathcal{L}^3)$ is another section on U satisfying the conditions of the proposition. Then

$$G = \lambda f^3 + \nu g$$

for some $\lambda \in \Gamma(U, (\mathcal{E}_3^+)^{-1} \otimes \mathrm{Sym}^3(\mathcal{E}_1))$ and $\nu \in \Gamma(U, \mathcal{O}_U)$, and there exists a section $A \in \Gamma(U, (\mathcal{E}_3^+)^{-2} \otimes \mathrm{Sym}^6(\mathcal{E}_1))$ such that $G^2 = Af^6$. By condition (i) in the proposition, $\nu \equiv 1$. So we have

$$\begin{aligned} G^2 &= \lambda^2 f^6 + \lambda f^3 g + g^2 \\ &= (\lambda^2 + a) f^6 + \lambda f^3 g \\ &= A f^6. \end{aligned}$$

The last equality holds if and only if $\lambda = 0$. So $G = g$, and g is unique. Patching these g together, we obtain the required global section of S_0. □

Putting all of this together, we have locally free sheaves $\mathcal{E}_1$ and $\mathcal{E}_3^+$ on S of ranks one and three respectively, and sections $f \in \Gamma(S_0, \mathcal{E}_1^\vee \otimes \pi_*\mathcal{L})$ and $g \in \Gamma(S_0, (\mathcal{E}_3^+)^{-1} \otimes \pi_*\mathcal{L}^3)$ such that on S_0:

$$\begin{aligned} (\pi_*\mathcal{L})|_{S_0} &= (\mathcal{E}_1|_{S_0}).f \\ (\pi_*\mathcal{L}^2)|_{S_0} &= \mathrm{Sym}^2(\mathcal{E}_1|_{S_0}).f^2 \\ (\pi_*\mathcal{L}^3)|_{S_0} &= \mathrm{Sym}^3(\mathcal{E}_1|_{S_0}).f^3 \oplus (\mathcal{E}_3^+|_{S_0}).g \\ &\vdots \\ (\pi_*\mathcal{L}^6)|_{S_0} &= \mathrm{Sym}^6(\mathcal{E}_1|_{S_0}).f^6 \oplus \big(\mathrm{Sym}^3(\mathcal{E}_1|_{S_0}) \otimes \mathcal{E}_3^+|_{S_0}\big).f^3 g. \end{aligned}$$

Furthermore, there exists a section $a \in \Gamma(S_0, (\mathcal{E}_3^+)^{-2} \otimes \mathrm{Sym}^6(\mathcal{E}_1))$ such that

$$g^2 = af^6$$

in $\Gamma(S_0, (\mathcal{E}_3^+)^{-2} \otimes \pi_*\mathcal{L}^6)$.

It follows from the properties of f and g above that the natural homomorphisms $\pi^*\pi_*(\mathcal{L}^n|_{X_0}) \to \mathcal{L}^n|_{X_0}$ are surjective for all $n > 0$. By Proposition 1.2.7 this induces a morphism $\mu_0\colon X_0 \longrightarrow \tilde{\mathbb{P}}_{S_0}(\mathcal{E}_1 \oplus \mathcal{E}_3^+)$ that has image $W_0 := W_0^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a)$. Furthermore, since $\mathcal{L}|_{X_0}$ is π-ample, μ_0 is birational onto its image and does not contract any curves, so must be an isomorphism. By construction and the surjectivity of $\pi^*\pi_*(\mathcal{L}|_{X_0}) \to \mathcal{L}|_{X_0}$, the inverse image $\mu_0^*\mathcal{O}_{W_0}(1) = \mathcal{L}|_{X_0}$. Finally, we may extend μ_0 to a birational map $\mu\colon X - \to W^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a)$. This completes the proof of Theorem 1.3.3. $\square$

Chapter 2

The Relative Log Canonical Model

2.1 The Relative Log Canonical Model

Whilst the K3-Weierstrass model has several advantages, ultimately it suffers from a major flaw. This lies in the fact that the map $\mu\colon X - \to W^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a)$ is not necessarily a morphism. This makes it difficult to control the singularities appearing in the fibres of $W^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a)$, and thwarts attempts to calculate its properties. To see why this is, consider the following example:

Example 2.1.1. Let $(X, \pi, \mathcal{L})$ be a threefold fibred by K3 surfaces of degree two over a curve S, and suppose that F is a unigonal fibre in X. By Example 1.1.5, F can be seen as a complete intersection

$$\{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z].$$

As noted in Example 1.1.5, F cannot be seen as a double cover of $\mathbb{P}^2$. So it is not immediately clear what its image under the map $\mu\colon X - \to W^{(2)}(\mathcal{E}_1, \mathcal{E}_3^+, a)$ will be. In fact, it will turn out that the restriction of μ to F always agrees with the natural projection $\mathbb{P}(1,1,1,2,3) - \to \mathbb{P}(1,1,1,3)$. We will illustrate this fact with a simple example; the general case follows by a similar, albeit slightly more complicated, argument.

So consider the following simple K3-fibration: let Δ denote the open complex unit disc $\{t \in \mathbb{C} : |t| < 1\}$ and let X be the threefold

$$\{z^2 - f_6(x_i, y) = t^2 y - f_2(x_i) = 0\} \subset \Delta \times \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z]$$

with natural projection $\pi \colon X \to \Delta$. Furthermore, assume that f_6 has been chosen such that the hypersurface

$$\{f_6(x_i, y) = 0\} \subset \mathbb{P}_{(1,1,1,2)}[x_1, x_2, x_3, y]$$

does not contain the point $(0:0:0:1)$.

Over a point t in the open set $\Delta^* := \Delta - \{0\}$, the fibre X_t of π is given by

$$X_t = \{z^2 - f_6(x_i, y) = t^2 y - f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z]$$

with $t \neq 0$. This is isomorphic to the K3 surface of degree two

$$\{t^6 z^2 - f_6(t x_i, f_2(x_i)) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

with isomorphism induced by the projection $\mathbb{P}(1,1,1,2,3) - \to \mathbb{P}(1,1,1,3)$.

The fibre X_0 of $\pi \colon X \to \Delta$ over $0 \in \Delta$ is a complete intersection

$$X_0 = \{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z].$$

As $(0:0:0:1) \notin \{f_6(x_i, y) = 0\}$, by Example 1.1.5, X_0 is a unigonal fibre. So $\pi \colon X \to \Delta$

is a threefold fibred by K3 surfaces of degree two with a unigonal fibre over $0 \in \Delta$. Let $W \subset \Delta \times \mathbb{P}(1,1,1,3)$ denote the K3-Weierstrass model of $\pi\colon X \to \Delta$, with projection $p\colon W \to \Delta$ and birational map $\mu\colon X - \to W$.

Let X^* denote the part of X over Δ^*. By Theorem 1.3.3, the restriction of μ to X^* is an isomorphism. So

$$\mu(X^*) = \{t^6 z^2 - f_6(tx_i, f_2(x_i)) = 0\} \subset \Delta^* \times \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z].$$

The fibre of $p\colon W \to \Delta$ over $0 \in \Delta$ is obtained by taking the limit of the above fibres as $t \to 0$.

As $(0:0:0:1) \notin \{f_6(x_i, y) = 0\}$, the coefficient of the y^3 term in f_6 is non-zero. Hence, the limit of the fibres of $\mu(X^*)$ as $t \to 0$ is the hypersurface

$$\mu(X_0) = \{(f_2(x_i))^3 = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z].$$

This is exactly the image of X_0 under the projection $\mathbb{P}(1,1,1,2,3) - \to \mathbb{P}(1,1,1,3)$.

We note that this fibre is both singular and not reduced (in the terminology of Section 2.3 we say that $p\colon W \to \Delta$ is not semistable). This leads to bad singularities in W and makes it very difficult to calculate its properties.

In order to solve this problem we will need to construct a better model. To find such a model we turn our attention away from elliptic fibrations and instead draw inspiration from the theory of surfaces fibred by curves of genus two. Whilst this may at first seem like an unusual route to take, it is actually quite logical. After all, a general curve of genus two can be expressed as a double cover of $\mathbb{P}^1$ ramified over six points or, in other words, a sextic in $\mathbb{P}(1,1,3)$. With this description, the parallels with our setup are immediately apparent.

Indeed, in his paper [Hor77] Horikawa constructs a model for genus two fibrations that is in many ways analogous to our K3-Weierstrass model, albeit using a different method to perform the construction itself. After doing so he experiences the same problem that we have: the map to his model is not necessarily a morphism. This leads to the appearance of highly singular fibres in his models, which make them difficult to work with.

Fortunately for us, Catanese and Pignatelli [CP06] find a way to improve Horikawa's model, and in doing so solve many of its problems. Their solution is to consider the relative canonical model of the genus two fibration and, using its built-in structure, they are able to find a way to construct it explicitly. In Chapter 4 we will attempt to emulate this construction for our threefolds fibred by K3 surfaces of degree two.

However, we immediately encounter a problem. If we attempt to define the relative canonical model in the usual way, using direct images of multiples of the canonical divisor K_X, the model we obtain will be of little practical use. This happens because the canonical divisor vanishes on the general fibres of our K3-fibration, so these fibres will be contracted when we proceed to the relative canonical model. To avoid this problem, rather than focusing our attention on the canonical divisor K_X, it will instead be beneficial to study a small perturbation $(K_X + H)$ for a suitably defined divisor H. This will give rise to a much better model, the *relative log canonical model*.

Fortunately for us, the study of varieties with slightly perturbed canonical classes is well developed. Indeed, such structures are thought to form the natural setting for running the higher dimensional minimal model program. The remainder of this section will be concerned with outlining some of the basic definitions and results from this theory that will be used in the coming chapters. Everything in this section may be found in more detail in [KM98].

We begin by defining the canonical sheaf for a normal variety X. Note that as such a space may be singular in codimension 2, the usual definition of ω_X as $\bigwedge^n \Omega_{X/\mathbb{C}}$ may give a sheaf that is not even reflexive, let alone locally free. Instead, following [Rei87, (1.5)], we define $\omega_X := j_*(\bigwedge^n \Omega_{X^0/\mathbb{C}})$, where $j\colon X^0 \hookrightarrow X$ denotes the inclusion of the smooth locus X^0 into X. By the results of [Rei87, 1.5], this sheaf is divisorial and equal to the reflexivisation $(\bigwedge^n \Omega_{X/\mathbb{C}})^{\vee\vee}$. Thus ω_X corresponds to a Weil divisor on X, which will henceforth be called K_X. We say that X is *Gorenstein* if X is Cohen-Macaulay and ω_X is locally free.

We can now define of the basic objects of our study:

Definition 2.1.2 [KM98, 2.25]. *A* log pair (X, H) *is a pair consisting of a normal variety X and a $\mathbb{Q}$-divisor H on X, such that $m(K_X + H)$ is Cartier for some $m > 0$.*

Let (X, H) be a log pair. We would like to be able to talk about how singular (X, H) is. In order to do this we will consider different resolutions of the singularities of (X, H).

Suppose that $f\colon Y \to X$ is a birational morphism from a normal variety Y. Let $\mathrm{Ex}(f) \subset Y$ denote the exceptional locus of f and $E_i \subset \mathrm{Ex}(f)$ the irreducible exceptional divisors. Then, by [KM98, 2.25], there exist rational numbers $a(E_i, X, H)$ such that $m \cdot a(E_i, X, H)$ are integers and

$$\mathcal{O}_Y\big(m(K_Y + f_+^{-1}H)\big) \cong f^*\mathcal{O}_X\big(m(K_X + H)\big) \otimes \mathcal{O}_Y\Big(\sum_i \big(m \cdot a(E_i, X, H)\big)E_i\Big).$$

It is important to note here that the numbers $a(E_i, X, H)$ do not depend upon the particular choice of morphism $f\colon Y \to X$.

Definition 2.1.3 [KM98, 2.28]. $a(E, X, H)$ *is called the* discrepancy of E *with respect to* (X, H). *The* discrepancy of the pair (X, H) *is given by*

$$\mathrm{discrep}(X, H) := \inf_E \{a(E, X, H) : E \text{ is an exceptional divisor over } X\},$$

where this infimum is taken over all the exceptional divisors E of all the birational morphisms $f\colon Y \to X$.

Note that we can extend the definition of discrepancy to encompass other irreducible divisors $E \subset X$. If $H = \sum_i a_i H_i$ for irreducible divisors H_i, we define $a(H_i, X, H) = -a_i$ and $a(E, X, H) = 0$ for any irreducible divisor $E \subset X$ that is different from the H_i. Using this and the notion of numerical equivalence of $\mathbb{Q}$-divisors, for $f\colon Y \to X$ as above we may write

$$K_Y + f_+^{-1}H \equiv f^*(K_X + H) + \sum_{E_i \text{ exceptional}} a(E_i, X, H)E_i, \quad \text{or}$$
$$K_Y \equiv f^*(K_X + H) + \sum_{E_i \text{ arbitrary}} a(E_i, X, H)E_i$$

The discrepancy provides us with a way to "measure" how singular the pair (X, H) is. We have the following definition:

Definition 2.1.4 [KM98, 2.34]. *The pair* (X, H) *is called*

$$\left.\begin{array}{c}\text{terminal}\\ \text{canonical}\\ \text{log canonical}\end{array}\right\} \text{ if } \operatorname{discrep}(X, H) \left\{\begin{array}{l} > 0,\\ \geq 0,\\ \geq -1.\end{array}\right.$$

If $H = 0$ *we say simply that* X *has* terminal (*resp.* canonical, log canonical) singularities.

This theory would be of little use if we could not explicitly calculate the discrepancy of a given pair (X, H). Fortunately, the following result will help us out:

Proposition 2.1.5 [KM98, Corollary 2.31]. *Let* (X, H) *be a log pair and write* H *as a sum* $H = \sum a_i H_i$ *of irreducible divisors* H_i. *Assume that* X *is smooth,* $\sum H_i$ *has simple normal crossings and* $a_i \leq 1$ *for every* i. *Then*

$$\operatorname{discrep}(X, H) = \min\left\{ \min_{\substack{i \neq j \\ H_i \cap H_j \neq \emptyset}} \{1 - a_i - a_j\}, \min_i \{1 - a_i\}, 1\right\}$$

With this in place we are ready to define the relative log canonical model for log pairs.

Definition 2.1.6 [KM98, 3.50]. *Let (X,H) be a log pair and $\pi\colon X \to S$ be a proper morphism to a normal variety S. A pair (X^c, H^c) sitting in a diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & X^c \arrow[dl, "\pi^c"] \\ & S & \end{tikzcd}$$

is called a relative log canonical model *of (X,H) over S if:*

(1) *π^c is proper,*

(2) *ϕ^{-1} has no exceptional divisors,*

(3) *$H^c = \phi_+ H$,*

(4) *$K_{X^c} + H^c$ is π^c-ample, and*

(5) *$a(E,X,H) \leq a(E,X^c,H^c)$ for every ϕ-exceptional divisor $E \subset X$.*

Unfortunately, given a log pair (X,H), this definition does not guarantee that a relative log canonical model for (X,H) exists. The next section will be devoted to proving that one does under certain assumptions on (X,H).

Remark 2.1.7. All of the results in this section continue to hold in the case where X is a normal complex analytic space. The only place where we have to be careful is when we refer to "the canonical divisor K_X", as this may not be well-defined for a normal complex analytic space. However, as we really only use the linear equivalence class of K_X we can safely reformulate everything in terms of the canonical sheaf ω_X instead, which is well-defined for normal complex analytic spaces. See [KM98, Sections 2.2 and 3.8] for full details.

2.2 Existence of the Relative Log Canonical Model

The aim of this section is to provide a set of conditions on a log pair (X, H) admitting a proper morphism $\pi\colon X \to S$ to a normal variety S that will ensure the existence of a relative log canonical model for (X, H) over S. We begin with a result that will prove central to this endeavour:

Proposition 2.2.1. *Let (X, H) be a pair consisting of a nonsingular complex variety (or complex manifold) X and a divisor H on X, and let $\pi\colon X \to S$ be a proper morphism to a nonsingular variety (or complex manifold) S. Suppose that*

$$\mathcal{R}(X, H) := \bigoplus_{n \geq 0} \pi_* \mathcal{O}_X(nK_X + nH)$$

is a sheaf of locally finitely generated $\mathcal{O}_S$-algebras and that

$$\dim \mathbf{Proj}_S \mathcal{R}(X, H) = \dim X.$$

Then the natural map $\phi\colon X - \to \mathbf{Proj}_S \mathcal{R}(X, H)$ is birational and the exceptional set $\mathrm{Ex}(\phi^{-1})$ *has codimension at least 2 in $\mathbf{Proj}_S \mathcal{R}(X, H)$.*

Remark 2.2.2. Whilst this result is certainly well-known (for example, see [Kol08, Exercises 113 and 114]), we have been unable to find a satisfactory proof in the literature. A proof is included here for completeness.

Proof. We note first that, as ϕ preserves the fibration structure, the question is local in the base S. With this in mind, let $U \subset S$ be an affine open set (or Stein space), and let $X_U := \pi^{-1}(U)$ be the open set in X above U. Let $\psi\colon \mathbf{Proj}_S \mathcal{R}(X, H) \to S$ denote the natural morphism. By definition of the **Proj** construction and properties of the direct image, the open set $\psi^{-1}(U) \subset \mathbf{Proj}_S \mathcal{R}(X, H)$ is given by

$$\begin{aligned}\psi^{-1}(U) &= \operatorname{Proj}\big(H^0(U, \mathcal{R}(X,H)|_U)\big)\\ &= \operatorname{Proj}\Big(\bigoplus_{n\geq 0} H^0(X_U, \mathcal{O}_{X_U}(nK_X + nH))\Big).\end{aligned}$$

With this in place, we use an argument based upon that in [BHPvdV04, Section VII.5] to show that we may assume that the algebra

$$\mathcal{R}(U) := \bigoplus_{n\geq 0} H^0(X_U, \mathcal{O}_{X_U}(nK_X + nH))$$

is generated in degree one. We begin by noting that as $\mathcal{R}(X,H)$ is locally finitely generated, $\mathcal{R}(U)$ is finitely generated, so there exists an integer m_0 such that $\mathcal{R}(U)$ is generated in degree m_0. Then for any integer m, by the m-uple embedding, the subring

$$\mathcal{R}^{(m)}(U) := \bigoplus_{n\geq 0} H^0(X_U, \mathcal{O}_{X_U}(mnK_X + mnH))$$

defines a projective variety $\operatorname{Proj}\mathcal{R}^{(m)}(U)$ that is isomorphic to $\operatorname{Proj}\mathcal{R}(U)$, with isomorphism induced by the inclusion $\mathcal{R}^{(m)}(U) \subset \mathcal{R}(U)$.

Next, define $\mathcal{R}^{[m]}(U)$ to be the subring of $\mathcal{R}^{(m)}(U)$ generated by the sections in $H^0(X_U, \mathcal{O}_{X_U}(mK_X + mH))$. Since $\mathcal{R}(U)$ is generated in degree m_0, if $m \geq m_0$ then $\mathcal{R}^{[m]}(U) \cong \mathcal{R}^{(m)}(U)$. So the log pluricanonical map

$$\phi_m \colon X_U - \to \operatorname{Proj}\mathcal{R}^{[m]}(U)$$

defined by the linear system $|mK_X + mH|$ has image isomorphic to $\operatorname{Proj}\mathcal{R}(U)$ if $m \geq m_0$.

Putting all of this together, without loss of generality we may assume that we are in the following situation: $\pi \colon X \to U$ is a proper morphism from a connected nonsingular complex space X to a nonsingular affine variety (or Stein space) U. Choose $m \geq m_0$, so that the graded algebra

$$\mathcal{R}^{(m)}(X,H) := \bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(mnK_X + mnH))$$

is generated in degree one. We wish to show that the map

$$\phi_m\colon X - \to Z := \operatorname{Proj} \mathcal{R}^{(m)}(X,H)$$

defined by the linear system $|mK_X + mH|$ is birational and that the exceptional set $\operatorname{Ex}(\phi_m^{-1})$ has codimension at least 2 in Z.

Let $f\colon \tilde{X} \to X$ denote the blow up of the base points of $|mK_X + mH|$ and let $\tilde{H} = f^*(mK_X + mH)$. Write $|\tilde{H}| = |M| + F$, where F is the fixed locus and $|M|$ has no base points or fixed components. Then consider the morphism

$$g\colon \tilde{X} \longrightarrow \tilde{Z} := \operatorname{Proj}\Big(\bigoplus_{n\geq 0} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nM))\Big)$$

defined by the linear system $|M|$. We aim to show that $\tilde{Z} \cong Z$.

For any $n \geq 0$ we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(nM) \longrightarrow \mathcal{O}_{\tilde{X}}(n\tilde{H}) \longrightarrow \mathcal{O}_{nF}(n\tilde{H}) \longrightarrow 0$$

which gives rise to an inclusion of cohomology groups

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nM)) \hookrightarrow H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(n\tilde{H}))\ .$$

Furthermore, since $|\tilde{H}| = |M| + F$ and F is fixed, every section of $\mathcal{O}_{\tilde{X}}(n\tilde{H})$ arises from a section of $\mathcal{O}_{\tilde{X}}(nM)$, so this map is an isomorphism. Next, by the Leray spectral sequence, we have an isomorphism

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(n\tilde{H})) \cong H^0(X, f_*\mathcal{O}_{\tilde{X}}(n\tilde{H})).$$

Finally, as f is surjective and has connected fibres, $f_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and so by the projection formula

$$f_*\mathcal{O}_{\tilde{X}}(n\tilde{H}) \cong \mathcal{O}_X(nmK_X + nmH).$$

Putting all of this together, for any $n \geq 0$ we get an isomorphism of cohomology groups

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nM)) \cong H^0(X, \mathcal{O}_X(nmK_X + nmH)) \tag{2.1}$$

that induces an isomorphism $\tilde{Z} \cong Z$. Furthermore, the morphism g agrees with the composition $\phi_m \circ f$, so the diagram

$$\begin{tikzcd} \tilde{X} \arrow[d, "f"'] \arrow[dr, "g"] & \\ X \arrow[r, dashed, "\phi_m"'] & Z \end{tikzcd}$$

commutes. This reduces our problem to one of showing that g is birational and contracts the exceptional set $\mathrm{Ex}(f) \subset \tilde{X}$ to a codimension 2 subset in Z.

Remark 2.2.3. Note that this part of the proof also shows that if the linear system $|mK_X + mH|$ is base point free for $m \geq m_0$, the map ϕ_m is a morphism over the open set U. This will prove to be useful to us later.

We begin by showing that g is birational. Using the assumption that $\dim X = \dim Z$, Stein factorisation [Uen75, Theorem 1.9] gives that g is generically finite. Then by [Uen75, Corollary 5.8] we have that, for a suitably large choice of m, the general fibre of the morphism g is connected. So the general fibre of the morphism g must be a point, and thus g is birational.

In order to show that g contracts the exceptional set of f, we begin by showing that all of the exceptional divisors in $\mathrm{Ex}(f)$ appear in the fixed part F of the linear system $|\tilde{H}|$. In order to show this, first consider the (not necessarily complete) linear system $f^*|mK_X+mH|$ given by pulling-back the members of the linear system $|mK_X+mH|$ by f. As f is the blow up of the base points of $|mK_X+mH|$, any member of $f^*|mK_X+mH|$ contains all of the divisors in the exceptional locus $\mathrm{Ex}(f)$. So these exceptional divisors are all contained in the fixed part of $f^*|mK_X+mH|$.

Next, note that we have an inclusion $f^*|mK_X+mH| \subset |\tilde{H}|$. Using this and the analysis of $f^*|mK_X+mH|$ above, we see that in order to show that all of the divisors in $\mathrm{Ex}(f)$ appear in F, it suffices to show that this inclusion is actually an equality. To see this, note first that the inclusion $f^*|mK_X+mH| \subset |\tilde{H}|$ corresponds to an inclusion in cohomology

$$H^0(X, \mathcal{O}_X(mK_X+mH)) \subset H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{H})).$$

But, by (2.1) above, these cohomology groups are in fact isomorphic. So we have $f^*|mK_X+mH| = |\tilde{H}|$ and all of the exceptional divisors in $\mathrm{Ex}(f)$ appear in F.

With this in place, to complete the proof of Proposition 2.2.1 it suffices to show that g contracts the fixed locus F of the linear system $|\tilde{H}|$ to a codimension 2 subset in Z. We base our argument on that used by Reid in the proof of [Rei80, Lemma 1.6].

Let E be a component of F. For any $n \geq 0$ we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(nM) \longrightarrow \mathcal{O}_{\tilde{X}}(nM+E) \longrightarrow \mathcal{O}_E(nM+E) \longrightarrow 0.$$

In the corresponding long exact sequence of cohomology, the map

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nM)) \longrightarrow H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nM+E))$$

is an isomorphism, as E is fixed in the linear system $|nM+E|$. Using this, from the long exact sequence of cohomology we get

$$0 \longrightarrow H^0(E, \mathcal{O}_E(nM+E)) \longrightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(nM)) \longrightarrow \cdots .$$

Our strategy then becomes clear. If we can show that $h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(nM))$, and with it $h^0(E, \mathcal{O}_E(nM+E))$, is bounded above by (const.)$n^{\dim Z - 2}$ for sufficiently large n, then $\dim(g(E)) \leq \dim Z - 2$ and E will be contracted by g.

To show that $h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(nM)) \leq$ (const.)$n^{\dim Z-2}$, we consider the Leray spectral sequence for g:

$$0 \longrightarrow H^1(Z, g_*\mathcal{O}_{\tilde{X}}(nM)) \longrightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(nM)) \longrightarrow H^0(Z, R^1g_*\mathcal{O}_{\tilde{X}}(nM)) \longrightarrow \cdots .$$

As g is the morphism corresponding to the linear system $|M|$, for each $n \geq 0$ the direct image $g_*\mathcal{O}_{\tilde{X}}(nM)$ is isomorphic to $\mathcal{O}_Z(n)$. So, by Serre vanishing [Har77, Theorem III.5.2] [BS76, Theorem IV.2.1], we have $H^1(Z, g_*\mathcal{O}_{\tilde{X}}(nM)) = 0$ for sufficiently large n. Finally, as $\tilde{X}$ is irreducible and g is a birational morphism, g is an isomorphism outside of a codimension 2 subset of Z, and so $R^1g_*\mathcal{O}_{\tilde{X}}(nM)$ is supported in codimension ≥ 2. Thus, $h^0(Z, R^1g_*\mathcal{O}_{\tilde{X}}(nM)) \leq$ (const.)$n^{\dim Z-2}$ for sufficiently large n and so, by the sequence above $h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(nM)) \leq$ (const.)$n^{\dim Z-2}$.

This completes the proof of Proposition 2.2.1. □

With this in place, we have the following corollary:

Corollary 2.2.4. *Let (X, H) and $\pi \colon X \to S$ satisfy the conditions of Proposition 2.2.1. Then*

$$(X^c, H^c) := \big(\mathbf{Proj}_S \mathcal{R}(X, H), \phi_+ H\big)$$

is a relative log canonical model of (X, H) over S.

Proof. By Proposition 2.2.1, conditions (1) and (2) in the definition of the relative log canonical model are satisfied by (X^c, H^c). Furthermore, condition (3) is satisfied by construction and condition (4) follows from the properties of the **Proj** construction. Thus, we only have to check condition (5): that $a(E, X, H) \leq a(E, X^c, H^c)$ for any ϕ-exceptional divisor $E \subset X$.

In order to prove this, we use an argument based upon that used to prove [KM98, Lemma 3.38]. Let Z be a normal variety with birational morphisms $f\colon Z \to X$ and $g\colon Z \to X^c$, such that the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & X^c \arrow[dl, "\pi^c"] \\ & S & \end{tikzcd}$$

commutes. Then for suitable $m > 0$ we have

$$mK_Z \sim f^*(mK_X + mH) + \sum_i \big(m \cdot a(E_i, X, H)\big)E_i,$$
$$mK_Z \sim g^*(mK_{X^c} + mH^c) + \sum_i \big(m \cdot a(E_i, X^c, H^c)\big)E_i,$$

where the sums run over all divisors $E_i \subset Z$. Dividing by m and subtracting we obtain a $\mathbb{Q}$-divisor

$$B := \sum_i \big(a(E_i, X^c, H^c) - a(E_i, X, H)\big)E_i \equiv f^*(K_X + H) - g^*(K_{X^c} + H^c).$$

Now, $g^*(K_{X^c} + H^c)$ is g-nef and $f^*(K_X + H).C \leq 0$ for all curves C contracted by g. So $-B$ is g-nef. Moreover, as $\phi_+ H = H^c$, the irreducible components of B must be g-exceptional. Thus, by [KM98, Lemma 3.39], B must be effective and so $a(E_i, X^c, H^c) - a(E_i, X, H) \geq 0$ for all divisors $E_i \subset Z$. In particular, this must be true for any ϕ-exceptional divisor $E \subset X$. $\square$

In light of this result, the algebra $\mathcal{R}(X, H)$ will henceforth be called the *relative log canonical algebra* of the pair (X, H).

Now that we know when a relative log canonical model of (X, H) over S exists, the next theorem shows that, under certain assumptions on the singularities of (X, H), it is unique:

Theorem 2.2.5 [KM98, Theorem 3.52]. *Let (X, H) be a log canonical pair admitting a proper morphism $\pi\colon X \to S$ to a normal variety S. Then, if one exists, a relative log canonical model (X^c, H^c) of (X, H) over S is unique and*

$$(X^c, H^c) := \big(\mathbf{Proj}_S \mathcal{R}(X, H), \phi_+ H\big).$$

Remark 2.2.6. Note that as the relative log canonical algebra depends only upon the sheaves $\mathcal{O}_X(nK_X + nH)$, the relative log canonical model defined by this algebra is independent of the divisor H chosen from within its linear equivalence class. Thus we may safely refer to the relative canonical model of the pair $(X, \mathcal{L})$, where $\mathcal{L}$ is a line bundle on the complex manifold X. Note, however, that we must be careful when referring to the discrepancy in this situation, as it is not necessarily constant across the divisors of a linear system (in fact it is lower semicontinuous, see [KM98, Corollary 2.33]).

By Corollary 2.2.4, in order to show that the relative log canonical model of a pair (X, H) exists we need to prove that the relative log canonical algebra $\mathcal{R}(X, H)$ is locally finitely generated. This will follow from a version of the base point free theorem, first proved by Ancona [Anc87, Theorem 3.3].

Proposition 2.2.7. *Let $\pi\colon X \to S$ be a proper surjective morphism from a complex manifold X to a nonsingular curve S. Let $\mathcal{L}$ be a line bundle on X that satisfies:*

(i) *$\mathcal{L}$ is π-nef;*

(ii) $\omega_X + \mathcal{L}$ *is* π*-nef; and*

(iii) *there exists a dense open subset* $S_0 \subset S$ *such that for any* $s \in S_0$*, the restriction* $\mathcal{L}|_{X_s}$ *of* $\mathcal{L}$ *to the fibre* X_s *over* s *is big for* $s \in S_0$*.*

Then the relative log canonical algebra $\mathcal{R}(X,\mathcal{L}) := \bigoplus_{m=0}^{\infty} \pi_*(\omega_X^m \otimes \mathcal{L}^m)$ *is locally finitely generated as an* $\mathcal{O}_S$*-algebra and the natural map* $\phi\colon X \to \mathbf{Proj}_S\mathcal{R}(X,\mathcal{L})$ *is a morphism.*

Proof. This proof is based upon the arguments used to prove [KM98, Theorem 3.11] and [KMM87, Theorem 3.3.1].

By [Anc87, Theorem 3.3], under the assumptions of the proposition there exists a positive integer m_0 such that, for any $m \geq m_0$, the natural map

$$\pi^*\pi_*(\omega_X^m \otimes \mathcal{L}^m) \longrightarrow \omega_X^m \otimes \mathcal{L}^m \tag{2.2}$$

is surjective.

We first use this to show that $\mathcal{R}(X,\mathcal{L})$ is locally finitely generated. Suppose $m \geq m_0$. Let $\phi_m\colon X \to \mathbb{P} := \mathbb{P}_S(\pi_*(\omega_X^m \otimes \mathcal{L}^m))$ be the morphism associated to the surjection (2.2), and let $\pi'\colon \mathbb{P} \to S$ denote the natural projection. Then $\phi_m^*\mathcal{O}_{\mathbb{P}}(1) \cong \omega_X^m \otimes \mathcal{L}^m$ and, using the projection formula

$$\begin{aligned}\pi_*\big((\omega_X \otimes \mathcal{L})^{jm+r}\big) &\cong \pi'_* \circ (\phi_m)_*\big((\omega_X \otimes \mathcal{L})^{jm+r}\big)\\ &\cong \pi'_*\big(\mathcal{O}_{\mathbb{P}}(j) \otimes (\phi_m)_*(\omega_X^r \otimes \mathcal{L}^r)\big).\end{aligned}$$

As $\mathcal{O}_{\mathbb{P}}(1)$ is π'-ample, $\mathcal{R}_{\mathbb{P}} := \bigoplus_{j=0}^{\infty} \pi'_*\mathcal{O}_{\mathbb{P}}(j)$ is a locally finitely generated $\mathcal{O}_S$-algebra and

$$\bigoplus_{j=0}^{\infty} \pi'_*\big(\mathcal{O}_{\mathbb{P}}(j) \otimes (\phi_m)_*(\omega_X^r \otimes \mathcal{L}^r)\big)$$

is a locally finitely generated $\mathcal{R}_{\mathbb{P}}$-module for every $0 \leq r < m$. Thus, $\mathcal{R}(X,\mathcal{L})$ is a locally finitely generated $\mathcal{O}_S$-algebra.

We conclude by showing that the map ϕ is a morphism over a neighbourhood of any point $s \in S$. Note that, by Lemma 1.3.4, the sheaf $\pi_*(\omega_X^m \otimes \mathcal{L}^m)$ is locally free on S, so we may find an affine neighbourhood $U \subset S$ of s such that $(\pi_*(\omega_X^m \otimes \mathcal{L}^m))|_U$ is a free $\mathcal{O}_U$-module. Then for $m \geq m_0$, the surjection (2.2) gives that $(\omega_X^m \otimes \mathcal{L}^m)|_{\pi^{-1}(U)}$ is generated by its global sections. But then the proof of Proposition 2.2.1 gives immediately that ϕ is a morphism (see Remark 2.2.3). $\square$

With this result in place, we are almost ready to begin studying the relative log canonical models of our threefolds fibred by K3 surfaces of degree two. However, before we can do this we need to check that they exist. This will follow from Corollary 2.2.4 and Proposition 2.2.7 if we can show that the conditions of these two theorems are satisfied. In order to do this, we will use techniques from the birational geometry of degenerations. These techniques, along with several others that will be used in subsequent chapters, will be outlined in the next section.

2.3 An Introduction to Degenerations

In this section, the object of our study will be a proper, flat, surjective morphism

$$\pi \colon X \longrightarrow \Delta := \{z \in \mathbb{C} : 0 \leq |z| < 1\},$$

whose general fibre $X_t := \pi^{-1}(t)$ for $t \in \Delta^* := \Delta - \{0\}$ is a nonsingular K3 surface. Such an object is called a *degeneration* of K3 surfaces. Note that this definition does not assume that X is algebraic.

We will be interested in studying the properties of the central fibre $X_0 = \pi^{-1}(0)$. At present, this interest only extends as far as the numerical properties of the interaction between this fibre and the polarisation divisor. However, in later chapters we will also be interested in its underlying structure. In order to perform this study, we first wish to find a "nice" model for the degeneration, whose central fibre has certain desirable properties.

Our point of departure will be the *semistable reduction theorem*. However, in order to state it we need to introduce the operation of *base change of order* m. Let $\pi\colon X \to \Delta$ be a degeneration of K3 surfaces. Then the degeneration $\pi'\colon X' \to \Delta$ obtained from π by base change of order m is defined by the pull-back:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ \Delta \arrow[r, "\sigma"] & \Delta \end{tikzcd}$$

where the map σ is defined by:

$$\sigma\colon t \longmapsto t^m.$$

We can now state the semistable reduction theorem, first proved by Knudsen, Mumford and Waterman:

Theorem 2.3.1 (Semistable Reduction) [KKMSD73]. *Given $\pi\colon X \to \Delta$, there exists an m such that, if $\pi'\colon X' \to \Delta$ is the base change of order m, there is a birational morphism $Y \to X'$ so that $\rho\colon Y \to \Delta$ is* semistable, *i.e. Y is nonsingular and $Y_0 = \rho^{-1}(0)$ is a reduced divisor with normal crossings.*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\rho"'] & X' \arrow[r] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"'] \\ \Delta \arrow[r, equal] & \Delta \arrow[r] & \Delta \end{tikzcd}$$

Thus, by performing a base change and some birational modifications, we may assume that our central fibre X_0 is semistable. This already gives us quite a lot of information. Write $X_0 = \bigcup V_i$, where the V_i are the irreducible components of X_0 and we assume that the V_i have been normalised. Let $D_{ij} := V_i \cap V_j$, where if i and j are equal this denotes the preimage of the self-intersection locus under the normalisation map; such D_{ij} are called *double curves*. The *double locus* of X_0 is defined to be the union of the double curves $D := \bigcup D_{ij}$ and coincides with the singular locus of X_0. Finally, a *triple point* is defined to be any point in D where three of the V_i intersect. Then, if $\pi\colon X \to \Delta$ is semistable, we have the classically known *triple point formula*:

$$(D_{ij}|_{V_i})^2 + (D_{ij}|_{V_j})^2 = -T_{ij}$$

where T_{ij} is the number of triple points lying on the component D_{ij} of D.

Whilst this is all well and good, we can do even better! After all, we have yet to use the the fact that the generic fibre in our degeneration is a K3 surface. With this in mind we have the following theorem, courtesy of Kulikov [Kul77] [Kul81] and Persson-Pinkham [PP81]:

Theorem 2.3.2. *If $\pi\colon X \to \Delta$ is semistable with $\omega_{X_t} \cong \mathcal{O}_{X_t}$ for all $t \in \Delta^*$, and if all components of X_0 are Kähler, then there exists a birational modification X' of X such that $\pi'\colon X' \to \Delta$ is semistable, isomorphic to X over Δ^* and has $\omega_{X'} \cong \mathcal{O}_X$.*

Motivated by this theorem, we define a *Kulikov model* to be a semistable degeneration of K3 surfaces $\pi\colon X \to \Delta$ with $\omega_X \cong \mathcal{O}_X$.

Remark 2.3.3. The construction of the Kulikov model given by Persson and Pinkham is very non-algebraic in nature, involving the contraction of components of the central fibre that are only “generically contractible”. This means that even if X is algebraic, we cannot guarantee that its Kulikov model will be. We know only that the Kulikov model is complex analytic, and that the components of its central fibre are Kähler. See [PP81] for full details.

Kulikov models have enough structure that it is possible to completely classify their central fibres. This has been done by Persson [Per77], Kulikov [Kul77] and Friedman-Morrison [FM83]. However in order to state their result we first need to introduce the *dual graph* of the central fibre of a degeneration:

Definition 2.3.4. *Let $X_0 = \bigcup V_i$ be the central fibre in a semistable degeneration. Define the dual graph Γ of X_0 as follows: Γ is a simplicial complex whose vertices $P_1, \ldots, P_r$ correspond to the components $V_1, \ldots, V_r$ of X_0; the k-simplex $\langle P_{i_0}, \ldots, P_{i_k} \rangle$ belongs to Γ if and only if $V_{i_0} \cap \cdots \cap V_{i_k} \neq \emptyset$.*

This enables us to state:

Theorem 2.3.5 (Classification of Kulikov models). *Let $\pi \colon X \to \Delta$ be a semistable degeneration of K3 surfaces with $\omega_X \cong \mathcal{O}_X$, such that all components of X_0 are Kähler. Then either*

(I) *X_0 is a smooth K3 surface;*

(II) *X_0 is a chain of elliptic ruled components with rational surfaces at each end, and all double curves are smooth elliptic curves;*

(III) *X_0 consists of rational surfaces meeting along rational curves which form cycles in each component. If Γ is the dual graph of X_0, then $|\Gamma|$, the topological support of Γ, is homeomorphic to the sphere S^2.*

A Kulikov model of a degeneration of K3 surfaces will be referred to as a *degeneration of Type I, II or III*, depending upon which case of the theorem it satisfies.

Unfortunately Kulikov models of degenerations are not unique. Examples of this are provided by the *elementary modifications of Types 0, I and II* [FM83], which map a given Kulikov model to a birationally equivalent one. We briefly review these here, as they will turn out to be important tools in subsequent chapters.

Elementary Modifications of Type 0. Suppose that $C \subset X_0$ is a smooth rational curve of self-intersection (-2) which does not meet the double locus of X_0. The curve C can be contracted on X_0 to give a surface $\overline{X_0}$ with an ordinary double point, and this

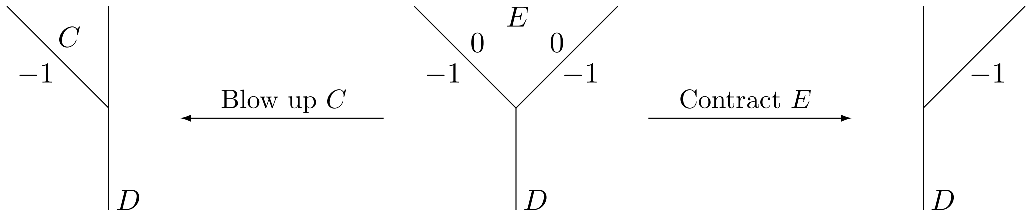

Figure 2.1: Elementary Modification of Type I.

contraction is induced by a birational modification with $\overline{X}$ singular:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ \Delta \arrow[r, equal] & \Delta \end{tikzcd}$$

We say that C extends to X_t if $\overline{X_t}$ has an ordinary double point for all $t \in \Delta^*$. If C does not extend to X_t, then the singular space $\overline{X}$ has a second, distinct resolution X':

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X' \arrow[dl] \\ & \overline{X} & \end{tikzcd}$$

In this case we call C a (§)-*curve*, and the induced birational map $X- \to X'$ is the *elementary modification of Type 0* along C.

Elementary Modifications of Type I. Suppose that $C \subset X_0$ is a smooth rational curve of self-intersection (-1), meeting a double curve D in one point; we call such C a (†)-*curve*. Begin by blowing up C in X. The exceptional divisor E is isomorphic to a copy of $\mathbb{P}^1 \times \mathbb{P}^1$, ruled in two different ways. Then contract E along the other ruling, moving the curve C to the neighbouring component. The resulting birational map is the *elementary modification of Type I* along C. This process is explained by Figure 2.1.

Elementary Modifications of Type II. Suppose that $C \subset X_0$ is a double curve

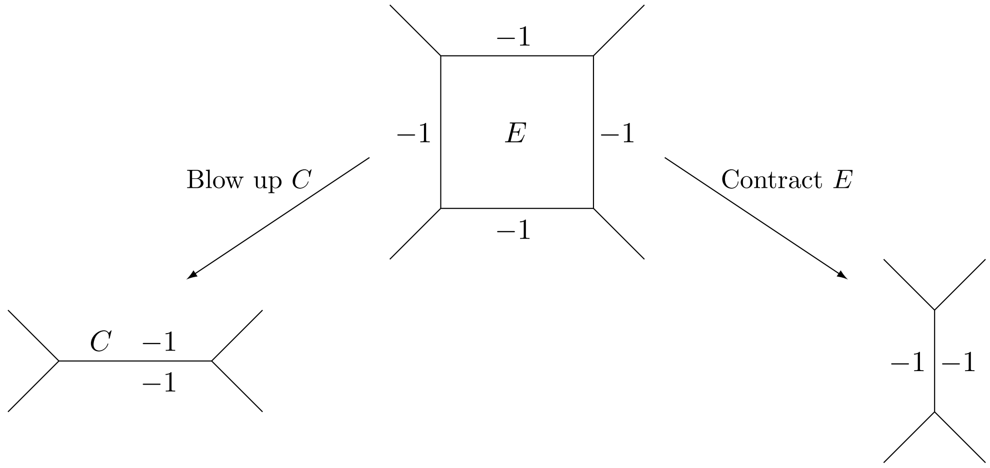

Figure 2.2: Elementary Modification of Type II.

which is smooth and rational, with self-intersection (-1) in both components in which it lies; we call such C a $(*)$-*curve*. Begin by blowing up C in X. The exceptional divisor E is again isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then contract E along the other ruling. The resulting birational map is the *elementary modification of Type II* along C. This process is explained by Figure 2.2.

These operations are important because of the following result, originally proved by Friedman and improved by Shepherd-Barron:

Theorem 2.3.6 [SB83b, Corollary 3.1]. *Suppose that* $\pi\colon X \to \Delta$ *and* $\pi'\colon X' \to \Delta$ *are semistable degenerations of algebraic K3 surfaces that are isomorphic over* Δ^* *and have* ω_X, $\omega_{X'}$ *trivial. Then the birational map* $f\colon X - \to X'$ *is a composition of elementary modifications.*

Elementary modifications are also important in the study of relative log canonical models, as we see from the following lemma:

Lemma 2.3.7. *Suppose that* $\pi_i\colon X_i \to \Delta$ *for* $i = 1, 2$ *are two semistable degenerations of K3 surfaces with* ω_{X_i} *trivial. Suppose further that* X_2 *is obtained from* X_1 *by an* *elementary modification $\psi\colon X_1 \dashrightarrow X_2$. Let H be an effective divisor on X_1. Then (X_1, H) and $(X_2, \psi_+ H)$ have the same relative log canonical algebra over Δ.*

Proof. We can find a nonsingular Y and morphisms f_1 and f_2 so that the diagram

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X_1 \arrow[rr, dashed, "\psi"] \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & \Delta & \end{tikzcd}$$

commutes (Y is a *resolution of indeterminacies* for ψ). Furthermore, as X_i is nonsingular and ω_{X_i} is trivial, $\omega_Y \cong \mathcal{O}_Y(E)$ for some E effective and f_i-exceptional for each i. Finally, noting that $f_1^*(H) = f_2^*(\psi_+ H)$, by [KM98, Corollary 3.53] the relative log canonical algebras of (X_1, H), $(X_2, \psi_+ H)$ and $(Y, f_1^* H)$ must all be equal. $\square$

Before we state the main theorem of this section, we need a definition:

Definition 2.3.8. *Let $\pi\colon X \to S$ be a proper, flat surjective morphism of nonsingular complex varieties (or complex manifolds) and let H be an effective divisor on X. We say that H is* π-flat *if $H \cap X_s$ is a well-defined divisor on each fibre X_s of $\pi\colon X \to S$.*

We are now in a position to state a theorem of Shepherd-Barron [SB83b, Theorem 1] that will enable us to prove the numerical conditions on the polarisation divisor that we require in order to apply Proposition 2.2.7. We split the original theorem into two parts for convenience:

Theorem 2.3.9 [SB83b, Theorem 1(a)]. *Suppose that $\pi\colon X \to \Delta$ is a semistable degeneration of K3 surfaces. Assume that $\mathcal{L} \in \mathrm{Pic}(X)$ induces a nef and big line bundle $\mathcal{L}_t \in \mathrm{Pic}(X_t)$ for all $t \in \Delta^*$. Then there is an effective or zero divisor Z supported on the components of X_0 such that $\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}_X(-Z)$ is of the form $\mathcal{L}' = \mathcal{O}_X(H)$, where H is effective and π-flat over Δ.*

Theorem 2.3.10 [SB83b, Theorem 1(b)]. *Suppose that $\pi\colon X \to \Delta$ is a semistable degeneration of K3 surfaces such that $\omega_X \cong \mathcal{O}_X$ (i.e. a Kulikov model). Assume that $\mathcal{L} \in \mathrm{Pic}(X)$ is a line bundle on X satisfying the conclusion of Theorem 2.3.9. Then a series of elementary modifications may be performed on X that transform $\mathcal{L}$ into a nef line bundle.*

With this result we are ready to begin studying relative log canonical models of threefolds fibred by K3 surfaces of degree two.

2.4 The Case of a Threefold Fibred by K3 Surfaces of Degree Two

In this section we return our attention to threefolds fibred by K3 surfaces of degree two. The aim is to emulate the local construction in the previous section, so that we may use Proposition 2.2.7 and Corollary 2.2.4 to prove the existence of their relative log canonical models.

Begin by letting S be a nonsingular curve. Let $(X, \pi, \mathcal{L})$ be a threefold fibred by K3 surfaces of degree two over S.

In order to apply the results of the last section, we would like to make the further assumption that the fibration $\pi\colon X \to S$ is semistable (i.e. all fibres of X are reduced divisors with normal crossings). We note that, by [BHPvdV04, Theorem III.10.3], a general threefold fibred by K3 surfaces of degree two can be transformed into a semistable one by pulling everything back to a finite cover of the base curve S, then resolving singularities.

This is all well and good but, unfortunately, if we wish to apply the theory developed in the last section to study the relative log canonical models produced (once we have proved their existence!) under this setup we run into a problem. The difficulty lies in the local assumptions that we made: specifically, in the fact that we did not assume algebraicity of the degeneration $\pi\colon X \to \Delta$. This means that the classification of Theorem 2.3.5 holds for certain non-algebraic degenerations, as well as for algebraic ones. Thus, if we wish to use this classification to characterise the relative log canonical models of threefolds fibred by K3 surfaces of degree two, we should expect our characterisation to contain gaps corresponding to the non-algebraic threefolds.

Fortunately, Shepherd-Barron provides us with a way to find a solution:

Theorem 2.4.1 [SB83b, Theorem 2(b)]. *Suppose that $\pi\colon X \to \Delta$ is a semistable degeneration of K3 surfaces such that $\omega_X \cong \mathcal{O}_X$ (i.e. a Kulikov model), and suppose that $\mathcal{L} \in \mathrm{Pic}(X)$ is a nef line bundle that is big on X_s for all $s \in \Delta^*$. Then there is an integer $N > 0$ and a divisor D supported on X_0 such that $\mathcal{M} := \mathcal{L}^N \otimes \mathcal{O}_X(D)$ defines a birational morphism $\phi^t\colon X \to X^t$ that contracts finitely many curves to Gorenstein terminal singularities.*

From this, we see that all of the analytic degenerations considered in the last section are birational to singular projective degenerations. Thus, we can fill in the gaps in our categorisation by considering singular threefolds fibred by K3 surfaces of degree two. We define:

Definition 2.4.2. *Let S be a nonsingular complex curve. A* terminal threefold fibred by K3 surfaces of degree two *over S, denoted $(X, \pi, \mathcal{L})$, consists of:*

(1) *A three dimensional complex variety X that has at worst Gorenstein terminal singularities;*

(2) *A projective, flat, surjective morphism $\pi\colon X \to S$ with connected fibres, whose general fibres are K3 surfaces;*

(3) *An invertible sheaf $\mathcal{L}$ on X that induces an ample invertible sheaf $\mathcal{L}_s$ with self-intersection number $\mathcal{L}_s.\mathcal{L}_s = 2$ on a general fibre X_s of $\pi\colon X \to S$.*

Next we need to define what it means for $\pi\colon X \to S$ to be semistable when X is singular.

Definition 2.4.3. *Let X be a normal complex variety with Gorenstein terminal singularities that admits a K3-fibration $\pi\colon X \to S$ over a nonsingular complex curve S. We say that $\pi\colon X \to S$ is* semistable *if*

(i) *all fibres of π are reduced divisors that have normal crossings outside of the singular locus on X,*

(ii) *for any singular point $p \in X$, let X_p be the fibre of π containing p. Then there exist an analytic neighbourhood $U_p \subset X$ of p and a local (analytic) resolution $f_p\colon Y_p \to U_p$ such that $\mathrm{Ex}(f_p)$ has codimension two in Y_p and $f_p^{-1}(X_p)$ has normal crossings.*

We remark that in the case where X is smooth, this notion coincides with the usual definition of semistability. We also note that, by definition, any semistable K3-fibration $\pi\colon X \to S$ admits an analytic resolution $f\colon \overline{X} \to X$ such that $\mathrm{Ex}(f)$ has codimension two in $\overline{X}$. We call such $f\colon \overline{X} \to X$ a *small analytic resolution* of X.

Remark 2.4.4. Gorenstein terminal singularities admitting small resolutions have been studied by several authors. Reid [Rei83, Corollary 1.12] has shown that any such singularity must be *compound Du Val* (see Definition 4.3.6). Furthermore, Katz [Kat91] has partially categorised which of the compound Du Val singularities admit small resolutions.

Note that the proper morphism $\overline{\pi} := \pi \circ f$ endows the complex manifold $\overline{X}$ with the structure of a semistable K3-fibration over S. In light of this, we will extend our definition of a threefold fibred by K3 surfaces of degree two in a second direction, to encompass the case where X is analytic.

Definition 2.4.5. *Let S be a nonsingular complex curve. An* analytic threefold fibred by K3 surfaces of degree two *over S, denoted $(X, \pi, \mathcal{L})$, consists of:*

(1) *A nonsingular compact complex threefold X;*

(2) *A proper, flat, surjective morphism $\pi\colon X \to S$ with connected fibres, whose general fibres are K3 surfaces;*

(3) *A line bundle $\mathcal{L}$ on X that induces an ample invertible sheaf $\mathcal{L}_s$ with self-intersection number $\mathcal{L}_s.\mathcal{L}_s = 2$ on a general fibre X_s of $\pi\colon X \to S$.*

Now that we have defined our objects of study, we would like to begin our study of their relative log canonical models. As before, let S be a nonsingular complex curve and let $(X, \pi, \mathcal{L})$ be a semistable terminal threefold fibred by K3 surfaces of degree two over S. We will be interested in studying the relative log canonical model of the pair $(X, \mathcal{L})$.

We remark that if $f\colon \overline{X} \to X$ is a small analytic resolution of X, [KM98, Corollary 3.53] shows that the relative log canonical algebras $\mathcal{R}(X, \mathcal{L})$ and $\mathcal{R}(\overline{X}, f^*\mathcal{L})$ agree so, if it exists, $(X, \mathcal{L})$ and $(\overline{X}, f^*\mathcal{L})$ have the same relative log canonical model over S. This means that we can use analytic threefolds fibred by K3 surfaces of degree two to study the relative log canonical models of terminal ones.

With this in place we would like to emulate the construction performed in Section 2.3 for a semistable terminal threefold fibred by K3 surfaces of degree two, so that we can use Proposition 2.2.7 and Corollary 2.2.4 to prove that its relative log canonical model exists.

In the style of Theorem 2.3.9, we would first like to find a twisted polarisation that agrees with $\mathcal{L}$ on a general fibre, but is better behaved on the singular fibres. We say:

Definition 2.4.6. *Let $\pi\colon X \to S$ be a proper, flat, surjective morphism of normal complex varieties (resp. compact complex manifolds), and suppose that $\mathcal{L}$ is an invertible sheaf (resp. line bundle) on X. Then we say that $\mathcal{L}$ is* locally π-flat *if for all closed points $s \in S$ there exists a neighbourhood U_s of s and a section in $\Gamma(\pi^{-1}(U_s), \mathcal{L})$ that defines an effective and π-flat divisor over U_s.*

Working on a small analytic resolution $f\colon \overline{X} \to X$ and looking locally around each of the singular fibres, using Theorem 2.3.9 we may find a divisor Z, supported on the components of the singular fibres of $\overline{\pi}\colon \overline{X} \to S$, so that the twisted invertible sheaf $f^*\mathcal{L} \otimes \mathcal{O}_{\overline{X}}(Z)$ is locally $\overline{\pi}$-flat on $\overline{X}$. Then as $\mathrm{Ex}(f)$ has codimension two in $\overline{X}$, the twisted sheaf $\mathcal{L}\otimes\mathcal{O}_X(f_+Z)$ must also be locally π-flat on X. We note that $\mathcal{L}\otimes\mathcal{O}_X(f_+Z)$ agrees with $\mathcal{L}$ on a general fibre of π, so $(X, \pi, \mathcal{L} \otimes \mathcal{O}_X(f_+Z))$ is a semistable terminal threefold fibred by K3 surfaces of degree two.

In light of this, for the remainder of this chapter we will assume that $(X, \pi, \mathcal{L})$ is a semistable terminal threefold fibred by K3 surfaces of degree two such that $\mathcal{L}$ is locally π-flat. We will denote by $f\colon \overline{X} \to X$ a small analytic resolution of X and define $\overline{\mathcal{L}} := f^*\mathcal{L}$. Note that as $\mathrm{Ex}(f)$ has codimension two, $\overline{\mathcal{L}}$ must also be locally $\overline{\pi}$-flat.

Given this, the rest of this section will be devoted to showing that the relative log canonical model of $(X, \mathcal{L})$ exists. In order to do this, we attempt to emulate the local construction given in Section 2.3. We begin by looking at the invertible sheaf $\mathcal{L}$ in more detail.

Proposition 2.4.7. *Suppose that $(X, \pi, \mathcal{L})$ is a semistable terminal threefold fibred by K3 surfaces of degree two over a nonsingular curve S. Then we may decompose $\mathcal{L}$ as $\mathcal{L} \cong \mathcal{O}_X(H) \otimes \pi^*\mathcal{M}$, where $\mathcal{M}$ is an invertible sheaf on S and H is an effective, irreducible and π-flat Cartier divisor on X.*

Proof. Note that in order to prove the proposition, it suffices to show that we may find an invertible sheaf $\mathcal{M}$ on S such that $\mathcal{L} \otimes \pi^*\mathcal{M}^{-1} \cong \mathcal{O}_X(H)$ for some effective, irreducible and π-flat divisor H.

To construct $\mathcal{M}$, we begin by choosing some ample invertible sheaf $\mathcal{N}$ on S. By the ampleness property, we may find an integer $m > 0$ such that $\pi_*\mathcal{L} \otimes \mathcal{N}^m$ is generated by its global sections. Furthermore, by the projection formula and the Leray spectral sequence, we have an isomorphism

$$H^0(X, \mathcal{L} \otimes \pi^*(\mathcal{N}^m)) \cong H^0(S, \pi_*\mathcal{L} \otimes \mathcal{N}^m). \tag{2.3}$$

In particular, the space of sections $H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m)$ is nonempty. Let D be an effective divisor defined by a general section in this space.

We make a brief digression here to explain what we mean by a *general section*. We say that an irreducible component D_i of an effective divisor D is *horizontal* if $\pi(D_i) = S$ and *vertical* if $\pi(D_i)$ is a closed point in S. Let D^h denote the sum of the horizontal components of D and D^v denote the sum of the vertical components. As π is proper, the image of any irreducible component must be closed and connected, so any irreducible component of D is either horizontal or vertical and $D = D^h + D^v$. Furthermore, D^h and D^v must be effective because D is.

Now let $s \in S$ be a point over which the fibre X_s is reducible and let V be any irreducible component of X_s. As $\mathcal{L}$ is locally π-flat and $\pi^*\mathcal{N}^m$ is a sum of fibres, $\mathcal{L} \otimes \pi^*\mathcal{N}^m$ must also be locally π-flat. So there exists a neighbourhood U_s of s and a section of $H^0(\pi^{-1}(U_s), \mathcal{L} \otimes \pi^*\mathcal{N}^m)$ that does not vanish on V. Then, since $\pi_*\mathcal{L} \otimes \mathcal{N}^m$ is generated by its global sections, using the isomorphism (2.3) we find that there exists a global section of $\mathcal{L} \otimes \pi^*\mathcal{N}^m$ that does not vanish on V. So the natural injection

$$i_V \colon H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m \otimes \mathcal{O}_X(-V)) \longrightarrow H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m)$$

cannot be surjective. Let U_V be the Zariski open set in $H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m)$ defined by the complement of the image of i_V. By taking the intersection of the U_V thus obtained over all of the (finitely many) components of all the reducible fibres, we obtain a nonempty open set $U \subset H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m)$. We say that D is defined by a general section if the section defining it is contained in U.

By construction, we see that only components of irreducible fibres may appear in D^v. So D^v must be a sum of fibres and as such can be written as the inverse image of an effective divisor E on S.

We have

$$\mathcal{O}_X(D^h) \cong \mathcal{L} \otimes \pi^*(\mathcal{N}^m \otimes \mathcal{O}_S(-E)).$$

Let $\mathcal{M} = \mathcal{N}^{-m} \otimes \mathcal{O}_S(E)$. In order to complete the proof of Proposition 2.4.7 we just need to show that D^h is irreducible and π-flat.

We begin with π-flatness. Let D_i^h denote a reduced and irreducible component of D^h. To show that D_i^h is π-flat, it suffices to show that D_i^h is flat when considered as a scheme over S. As S is a nonsingular curve, by [Har77, Proposition III.9.7] this will follow if we can show that any associated point of D_i^h maps to the generic point of S. But D_i^h is reduced and irreducible, so its only associated point is the generic point, which maps to the generic point of S as $\pi|_{D_i^h}$ is surjective. Thus every component D_i^h of D^h is π-flat, so D^h must be also.

Finally, we have to show that D^h is irreducible. As the restriction of $\mathcal{L} \otimes \pi^*\mathcal{N}^m$ to a general fibre of π defines an ample invertible sheaf with self-intersection number two, it is easy to see that the intersection of D^h with any such fibre is an irreducible curve of genus two. Thus, since D^h is π-flat and irreducible on a general fibre, it must be irreducible. $\square$

Let H and $\mathcal{M}$ be defined as in Proposition 2.4.7. Note that, as the exceptional set $\mathrm{Ex}(f)$ has codimension two, the decomposition of $\mathcal{L}$ given by this proposition induces a decomposition $\overline{\mathcal{L}} \cong \mathcal{O}_{\overline{X}}(f_+^{-1}H) \otimes \overline{\pi}^*\mathcal{M}$ on $\overline{X}$. With this in mind, we define $\overline{H} := f_+^{-1}H$.

Our next step will be to find a new space birational to X that shares the same relative log canonical model but has certain desirable properties that make it easier to study. This will be the analogue of the Kulikov model studied in the last section.

As $\overline{\pi}\colon \overline{X} \to S$ is semistable and the components of its fibres are projective (so Kähler), by Theorem 2.3.2 we may find a birational modification g that fits into a commutative diagram

$$\begin{tikzcd} \overline{X} \arrow[rr, dashed, "g"] \arrow[dr, "\overline{\pi}"'] & & X' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

where X' is a nonsingular compact complex manifold, all fibres of π' are semistable and any closed point $s \in S$ has an open neighbourhood U_s such that $\omega_{\pi'^{-1}(U_s)} \cong \mathcal{O}_{\pi'^{-1}(U_s)}$. Such X' is called a *locally Kulikov model*. Note that f only modifies finitely many fibres, as $\omega_{\overline{\pi}^{-1}(U_s)} \cong \mathcal{O}_{\overline{\pi}^{-1}(U_s)}$ is already satisfied if the fibre X_s of π over s is nonsingular.

Next we need to define a polarisation on X'. We have:

Proposition 2.4.8. *With notation as above, define $\mathcal{L}' \cong \mathcal{O}_{X'}(g_+\overline{H}) \otimes (\pi')^*\mathcal{M}$. Then $\mathcal{L}'$ is a locally π'-flat line bundle and $(X', \pi', \mathcal{L}')$ is a semistable analytic threefold fibred by K3 surfaces of degree two. Furthermore, the pairs $(X', \mathcal{L}')$ and $(\overline{X}, \overline{\mathcal{L}})$ define the same relative log canonical algebra over S.*

Proof. Note first that $\mathcal{L}'$ must be locally π'-flat, as $\mathrm{Ex}(g^{-1})$ has codimension two in X'. Furthermore, as g is an isomorphism on a general fibre of $\overline{\pi}$, the restriction of $\mathcal{L}'$ to such a fibre agrees with the restriction of $\overline{\mathcal{L}}$, so must define an ample line bundle with self-intersection number two. Thus $(X', \pi', \mathcal{L}')$ is a semistable analytic threefold fibred by K3 surfaces of degree two.

To show that the relative log canonical algebras agree, we will begin by considering the relative log canonical algebras of $(\overline{X}, \overline{H})$ and $(X', g_+\overline{H})$. As $\overline{H}$ is irreducible and $\overline{\pi}$-flat, the strict transform $g_+\overline{H}$ must also be irreducible and π'-flat. Thus, as $\overline{X}$ and X' are smooth, Proposition 2.1.5 gives $\mathrm{discrep}(\overline{X}, \overline{H}) = 0$ and $\mathrm{discrep}(X', g_+\overline{H}) = 0$, so the log pairs $(\overline{X}, \overline{H})$ and $(X', g_+\overline{H})$ are both canonical.

Now let

$$\begin{tikzcd} & Y \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ \overline{X} \arrow[rr, dashed, "g"] & & X' \end{tikzcd}$$

be a resolution of indeterminacies for the map g. Then as $(\overline{X}, \overline{H})$ is canonical, by definition

$$\omega_Y \otimes \mathcal{O}_{\overline{X}}\left(g_{1\,+}^{-1}\overline{H}\right) \cong g_1^*\left(\omega_{\overline{X}} \otimes \mathcal{O}_{\overline{X}}(\overline{H})\right) \otimes \mathcal{O}_Y(E_1),$$

for some effective and g_1-exceptional E_1. So, by [KM98, Corollary 3.53], the relative log canonical algebras of $(\overline{X}, \overline{H})$ and $(Y, g_{1\,+}^{-1}\overline{H})$ are isomorphic. Similarly, the relative log canonical algebras of $(X', g_+\overline{H})$ and $(Y, g_{2\,+}^{-1}g_+\overline{H})$ are also isomorphic.

Thus, for all $n > 0$ we have isomorphisms

$$\begin{aligned}\overline{\pi}_*\left(\omega_{\overline{X}}^n \otimes \mathcal{O}_{\overline{X}}(\overline{H})^n\right) &\cong (\overline{\pi} \circ g_1)_*\left(\omega_Y^n \otimes \mathcal{O}_Y(g_{1\,+}^{-1}\overline{H})^n\right)\\ &\cong (\pi' \circ g_2)_*\left(\omega_Y^n \otimes \mathcal{O}_Y(g_{2\,+}^{-1}g_+\overline{H})^n\right)\\ &\cong \pi'_*\left(\omega_{X'}^n \otimes \mathcal{O}_{X'}(g_+\overline{H})^n\right).\end{aligned}$$

Tensoring both sides by $\mathcal{M}^n$ and applying the projection formula we obtain

$$\overline{\pi}_*(\omega_{\overline{X}}^n \otimes \overline{\mathcal{L}}^n) \cong \pi'_*(\omega_{X'}^n \otimes \mathcal{L}'^n)$$

for all $n > 0$. This completes the proof. □

Thus to recap, we may assume that we are in the following situation: X' is a threefold that admits a K3-fibration $\pi' \colon X' \to S$, with X' locally Kulikov. Note that X' may be non-algebraic, but that the irreducible components of the fibres of $\pi' \colon X' \to S$ are Kähler. $\mathcal{L}'$ is a locally π'-flat line bundle on X' making $(X', \pi', \mathcal{L}')$ into a semistable analytic threefold fibred by K3 surfaces of degree two. Furthermore, if it exists, $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ define the same relative log canonical model over S.

Our final step is to prove an analogue of Theorem 2.3.10: we show that we can perform a series of elementary modifications to X' to obtain a new pair $(X'', \mathcal{L}'')$ with the desirable properties that its relative log canonical model X^c is well-defined and the birational map $X'' - \to X^c$ is actually a morphism. Furthermore, as we will use only elementary modifications on finitely many fibres to reach X'', the triple $(X'', \pi'', \mathcal{L}'')$ will define a semistable analytic threefold fibred by K3 surfaces of degree two and the pairs $(X'', \mathcal{L}'')$ and $(X', \mathcal{L}')$ will define the same relative log canonical model over S. So we may use $(X'', \mathcal{L}'')$ to study the relative log canonical model of $(X, \mathcal{L})$, as we originally intended.

The existence of $(X'', \mathcal{L}'')$ is given by the following theorem:

Theorem 2.4.9. *Let $(X', \pi', \mathcal{L}')$ be a semistable analytic threefold fibred by K3 surfaces of degree two over a nonsingular curve S. Suppose that X' is locally Kulikov and $\mathcal{L}'$ is locally π'-flat. Then there exists a birational transformation*

$$\begin{tikzcd} X' \arrow[rr, dashed, "\eta"] \arrow[dr, "\pi'"'] & & X'' \arrow[dl, "\pi''"] \\ & S & \end{tikzcd}$$

such that:

(i) *η is a composition of elementary modifications on finitely many fibres of π',*

(ii) *η transforms $\mathcal{L}'$ into a π''-nef and locally π''-flat line bundle $\mathcal{L}''$ on X'',*

(iii) *$X^c := \mathbf{Proj}_S \mathcal{R}(X'', \mathcal{L}'')$ is the relative log canonical model for X'' and the natural map $\phi'' \colon X'' \to X^c$ is a birational morphism.*

Proof. Let $S_0 \subset S$ be the Zariski open set of points over which the fibres of π' are smooth K3 surfaces that have an ample invertible sheaf with self-intersection number two induced upon them by $\mathcal{L}'$. The complement $S - S_0$ is a finite set of points; denote them by $s_1, \ldots, s_k$. Let X'_{s_i} denote the fibre over the point s_i.

Looking locally around each of the s_i, by Theorem 2.3.10, we may find a series of elementary modifications on X'_{s_i} that transforms $\mathcal{L}'$ into a π'-nef sheaf $\mathcal{L}''$ in a neighbourhood of X'_{s_i}. Let $\eta \colon X' - \to X''$ be the composition of the elementary modifications on all of the X'_{s_i}. Then as η only modifies a codimension 2 subset, $\mathcal{L}''$ is locally π''-flat. So, by construction, η and $\mathcal{L}''$ satisfy parts (i) and (ii) of the theorem.

To prove part (iii), we will first use Proposition 2.2.7 to show that the relative log canonical algebra

$$\mathcal{R}(X'', \mathcal{L}'') = \bigoplus_{n \geq 0}^{\infty} \pi''_*\big(\omega_{X''}^n \otimes (\mathcal{L}'')^n\big)$$

is locally finitely generated as an $\mathcal{O}_S$-algebra and the map $\phi''\colon X'' \to \mathbf{Proj}_S \mathcal{R}(X'', \mathcal{L}'')$ is a morphism. Then we will use Corollary 2.2.4 to show that ϕ'' is birational and $X^c = \mathbf{Proj}_S \mathcal{R}(X'', \mathcal{L}'')$ is the relative log canonical model for $(X'', \mathcal{L}'')$.

By the argument above, we know that $\mathcal{L}''$ is π''-nef on X''. Furthermore, since X'' is locally Kulikov, $\omega_{X''} \otimes \mathcal{L}''$ is also π''-nef. Finally, for any $s \in S_0$ the restriction $\mathcal{L}''_s$ is ample, so in particular it must be big. Hence, we may apply Proposition 2.2.7 to see that $\mathcal{R}(X'', \mathcal{L}'')$ is locally finitely generated as an $\mathcal{O}_S$-algebra and the natural map $\phi''\colon X'' \to \mathbf{Proj}_S \mathcal{R}(X'', \mathcal{L}'')$ is a morphism.

Before we can apply Corollary 2.2.4, we still have to show that the dimension of $\mathbf{Proj}_S \mathcal{R}(X'', \mathcal{L}'')$ is equal to three. In order to see this, we consider the restriction of $\mathbf{Proj}_S \mathcal{R}(X'', \mathcal{L}'')$ to S_0. Since S_0 is open in S, this restriction is isomorphic to $\mathbf{Proj}_{S_0} \mathcal{R}(\pi''^{-1}(S_0), \mathcal{L}''|_{\pi''^{-1}(S_0)})$. But $\mathcal{L}''$ is π''-ample over S_0, so the natural map

$$\pi''^{-1}(S_0) \longrightarrow \mathbf{Proj}_{S_0} \mathcal{R}\big(\pi''^{-1}(S_0), \mathcal{L}''|_{\pi''^{-1}(S_0)}\big)$$

is actually an isomorphism. Hence, the dimension of $\mathbf{Proj}_S \mathcal{R}(X'', \mathcal{L}'')$ is equal to the dimension of X'', which is three. So we may apply Corollary 2.2.4 to prove part (iii) of the theorem. $\square$

For future reference, the threefolds constructed in this section fit together into a diagram as shown in Figure 2.3. Note that in this diagram $\overline{X}$, X' and X'' (with the relevant polarisations) are all semistable analytic threefolds fibred by K3 surfaces of

$$
\begin{tikzcd}
\overline{X} \arrow[r, dashed, "g"] \arrow[d, "f"'] & X' \arrow[r, dashed, "\eta"] \arrow[dd, "\pi'"] & X'' \arrow[d, "\phi''"] \\
X \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & X^c \arrow[dl, "\pi^c"] \\
& S &
\end{tikzcd}
$$

Figure 2.3.

degree two and X is a semistable terminal threefold fibred by K3 surfaces of degree two.

We conclude this section with a result that will allow us to study the fibres of the relative log canonical model in the next chapter:

Corollary 2.4.10. *Given $\pi'' \colon X'' \to S$ and $\mathcal{L}''$ on X'' that satisfy the conclusions of Theorem 2.4.9, for all $i > 0$ and all $n > 0$ we have*

$$R^i\pi_*\big(\omega_{X''}^n \otimes (\mathcal{L}'')^n\big) = 0.$$

Proof. This will follow immediately from [Anc87, Theorem 2.1] if we can show that $\omega_{X''}^{n-1} \otimes (\mathcal{L}'')^n$ is π''-nef and that there exists a dense open subset S_0 of S such that $(\omega_{X''}^{n-1} \otimes (\mathcal{L}'')^n)|_{X''_s}$ is big for all fibres X''_s over closed points $s \in S_0$. However, these properties follow from the corresponding properties for $\mathcal{L}''$ (proved above) and the fact that $\omega_{X''}$ is trivial in a neighbourhood of any fibre (by the locally Kulikov assumption). $\square$

Chapter 3

Fibres of the Relative Log Canonical Model

3.1 Semi Log Canonical Surface Singularities

Before embarking on the main work of this chapter, we begin with a slight digression on semi log canonical surface singularities. These generalise the notion of log canonical singularities defined in Section 2.1. More information about them can be found in [KSB88, Section 4].

Intuitively, a surface with semi log canonical singularities is allowed to contain some double curves alongside the usual log canonical surface singularities. This property will make semi log canonical singularities very important when we come to classify the singularities occurring in the singular fibres of our relative log canonical models, as we expect degenerate fibres of Types II and III to contain some double curves.

Note first that, in this section, a *surface* will be defined to be a 2-dimensional reduced separated scheme of finite type over $\mathbb{C}$. In particular, we do not assume that our surfaces are normal or even irreducible. We call a surface *semi-smooth* if it is singular along certain double curves and smooth elsewhere. Formally, we define:

Definition 3.1.1 [KSB88, 4.2]. *A surface X will be called* semi-smooth *if every closed point $x \in X$ is either smooth or analytically isomorphic to one of*

(i) *A double normal crossing point $0 \in \{xy = 0\} \subset \mathbb{C}^3$; or*

(ii) *A pinch point $0 \in \{x^2 = zy^2\} \subset \mathbb{C}^3$.*

The singular locus of a semi-smooth surface X is a smooth curve D_X, which will be called the double curve of X*.*

This leads naturally to the concept of a *semi-resolution*:

Definition 3.1.2 [KSB88, 4.3]. *A map $f : Y \to X$ is called a* semi-resolution *of X if the following conditions are satisfied:*

(i) *f is proper;*

(ii) *Y is semi-smooth;*

(iii) *if D_Y is the double curve of Y, then no component of D_Y is mapped to a point;*

(iv) *there is a finite set $\Sigma \subset X$ such that $f : f^{-1}(X - \Sigma) \to X - \Sigma$ is an isomorphism.*

If $f : Y \to X$ is a semi-resolution, we say that a curve $E_i \subset Y$ is *exceptional* if $f(E_i)$ is a point. Let $E = \bigcup_i E_i$ be all the exceptional curves. Then we say that the semi-resolution $f : Y \to X$ is *good* if $E \cup D_Y$ has smooth components and transverse intersections. The next proposition says that good semi-resolutions exist:

Proposition 3.1.3 [KSB88, Proposition 4.5]. *Let X be a surface such that all but finitely many points of X are smooth or double normal crossing points. Then X has a good semi-resolution.*

Now we are ready to define a semi log canonical surface singularity. The definition closely mirrors that of a log canonical singularity (2.1.4), but we have to be more careful when dealing with the canonical sheaf ω_X as X may not be normal.

So let $x \in X$ be a surface singularity, such that $X - x$ is semi-smooth. As in the normal case, we define ω_X as the reflexivisation of the top wedge of the sheaf of differentials, $\omega_X := (\bigwedge^2 \Omega_X)^{\vee\vee}$. Then for any $m > 0$, define the reflexivised mth tensor power $\omega_X^{[m]} := (\omega_X^m)^{\vee\vee}$.

Suppose further that X is $\mathbb{Q}$-Gorenstein (i.e. X is Cohen-Macaulay and $\omega_X^{[m]}$ is locally free for some $m > 0$). Let $f\colon Y \to X$ be a good semi-resolution of X. Then we can write

$$\omega_Y^{[m]} \cong f^*\omega_X^{[m]} \otimes \mathcal{O}_Y\Big(\sum_i m a_i E_i\Big),$$

where the E_i are exceptional divisors and $a_i \in \mathbb{Q}$. Then:

Definition 3.1.4 [KSB88, 4.17]. *$x \in X$ as above is called a* semi log canonical singularity *if $a_i \geq -1$ for all i.*

Note that, by [KSB88, Remark 4.18], this notion is independent of the good semi-resolution chosen.

Now that we have defined them, we would like to have a classification of the possible semi log canonical singularities. Fortunately this has already been solved. We have:

Theorem 3.1.5 [KSB88, Theorem 4.21]. *Let $x \in X$ be a surface singularity, such that $X - x$ is semi-smooth. Suppose further that X is Gorenstein. Then $x \in X$ is semi log canonical iff it is locally analytically isomorphic to one of:*

- *A smooth point.*
- *A rational double point $0 \in \{z^2 = f(x, y)\} \subset \mathbb{C}^3$, where the branch curve $\{f(x, y) = 0\} \subset \mathbb{C}^2$ has an A-D-E singularity at $0 \in \mathbb{C}^2$.*
- *A double normal crossing point $0 \in \{xy = 0\} \subset \mathbb{C}^3$.*
- *A pinch point $0 \in \{x^2 = zy^2\} \subset \mathbb{C}^3$.*
- *A simple elliptic singularity.*

- *A cusp.*
- *A degenerate cusp.*

The remainder of this section will be devoted to a more detailed examination of simple elliptic singularities, cusps and degenerate cusps. These singularities can be classified by the form of their minimal semi-resolutions so, in order to understand them, we need to know what it means for a semi-resolution to be *minimal*:

Definition 3.1.6 [KSB88, 4.9]. *Let X be a surface that is semi-smooth outside of a finite number of points. $f\colon Y \to X$ be a semi-resolution and let $g\colon \overline{Y} \to Y$ be the normalisation of Y. Let $E = \bigcup_i E_i$ be the union of the exceptional curves. Then $f\colon Y \to X$ is called* minimal *if for all E_i the normalisation $\overline{E}_i = g^{-1}(E_i)$ is not a rational (-1)-curve.*

Note that, by [KSB88, Proposition 4.10], the minimal semi-resolution exists and is unique. However, it may not necessarily be a good semi-resolution.

The notion of minimal semi-resolution allows us to give a definition of the simple elliptic singularities, cusps and degenerate cusps:

Definition 3.1.7 [KSB88, 4.20]. *A Gorenstein surface singularity is called:*

- *Simple elliptic if it is normal and the exceptional divisor of the minimal resolution is a smooth elliptic curve.*
- *A cusp if it is normal and the exceptional divisor of the minimal resolution is a cycle of smooth rational curves or a rational nodal curve.*
- *A degenerate cusp if it is not normal and the exceptional divisor of the minimal semi-resolution is a cycle of smooth rational curves or a rational nodal curve.*

We conclude this section with a closer look at each of these singularities. The simplest class is that of the simple elliptic singularities. We will be interested in two types of these singularities, which Saito [Sai74] calls $\tilde{E}_7$ and $\tilde{E}_8$. We have:

Proposition 3.1.8. *Let $x \in X$ be a simple elliptic singularity. Let $f\colon Y \to X$ be a minimal resolution with smooth elliptic exceptional curve E.*

(i) *If $E^2 = -1$, then $x \in X$ is locally analytically isomorphic to*

$$\tilde{E}_8\colon \quad 0 \in \{z^2 = y(y-x^2)(y-\lambda x^2)\} \subset \mathbb{C}^3$$

for some $\lambda \in \mathbb{C} - \{0,1\}$.

(ii) *If $E^2 = -2$, then $x \in X$ is locally analytically isomorphic to*

$$\tilde{E}_7\colon \quad 0 \in \{z^2 = xy(y-x)(y-\lambda x)\} \subset \mathbb{C}^3$$

for some $\lambda \in \mathbb{C} - \{0,1\}$.

Proof. By the classification in [Lau77, Section V], simple elliptic singularities having exceptional curve E with $-E^2$ sufficiently small are determined up to change of coordinates by E^2 and the j-invariant of E. Moreover, [Sai74, Satz 1.9] implies that if $E^2 = -1$, then $x \in X$ is locally analytically isomorphic to

$$0 \in \{z^2 = y(y-x^2)(y-\lambda x^2)\} \subset \mathbb{C}^3$$

and if $E^2 = -2$, then $x \in X$ is locally analytically isomorphic to

$$0 \in \{z^2 = xy(y-x)(y-\lambda x)\} \subset \mathbb{C}^3,$$

for some $\lambda \in \mathbb{C} - \{0,1\}$. Furthermore, in both cases λ completely determines the j-invariant of E, given by

$$j(E) = \frac{4}{27}\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}.$$

$\square$

Note that both types of singularity mentioned in the proposition arise from double covers of $\mathbb{C}^2$ ramified over singular curves. With this in mind, we call a curve singularity

$$0 \in \{y(y - x^2)(y - \lambda x^2) = 0\} \subset \mathbb{C}^2$$

a *consecutive triple point* and

$$0 \in \{xy(y - x)(y - \lambda x) = 0\} \subset \mathbb{C}^2$$

a *quadruple point.*

The cusp singularities are somewhat more complex. Let $x \in X$ be a cusp singularity. Let $f\colon Y \to X$ be a minimal resolution, with exceptional divisor E. By definition, E is a cycle of smooth rational curves or a rational nodal curve. We study two possible types of cycle. The first, called an (n, r)-cycle (for $n < 0$, $r \geq 0$), consists of a cycle of $r + 1$ rational curves $E_0, \ldots, E_r$ satisfying

$$E_i.E_j = \begin{cases} n & \text{if } i = j = 0 \text{ and } r = 0, \\ n - 2 & \text{if } i = j = 0 \text{ and } r > 0, \\ -2 & \text{if } i = j \neq 0 \\ 1 & \text{if } i = j \pm 1 \pmod{r+1} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the rational nodal curve is a special case of the (n, r)-cycle, where $r = 0$. This configuration is illustrated in Figure 3.1.

The second, called an (n_1, n_2, r_1, r_2)-cycle (for $n_1, n_2 < 0$, $r_1, r_2 \geq 0$) consists of a

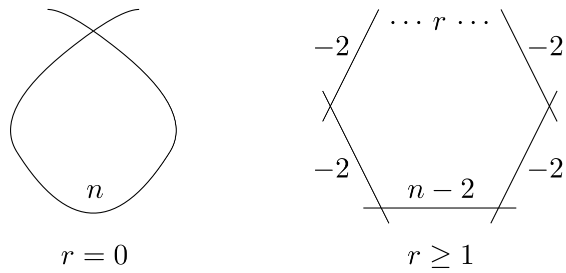

Figure 3.1: (n, r)-cycle.

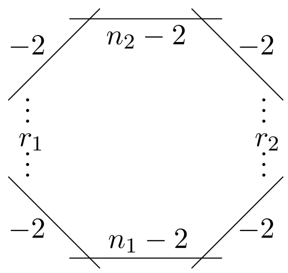

Figure 3.2: (n_1, n_2, r_1, r_2)-cycle.

cycle of $s = r_1 + r_2 + 2$ rational curves $E_0, \ldots, E_{s-1}$ satisfying

$$E_i.E_j = \begin{cases} n_1 - 2 & \text{if } i = j = 0 \\ n_2 - 2 & \text{if } i = j = r_1 + 1 \\ -2 & \text{if } i = j \notin \{0, r_1 + 1\} \\ 1 & \text{if } i = j \pm 1 \pmod{s} \\ 0 & \text{otherwise.} \end{cases}$$

This configuration is illustrated in Figure 3.2.

Using this notation, we have:

Proposition 3.1.9. *Let $x \in X$ be a cusp singularity. Let $f \colon Y \to X$ be a minimal resolution with exceptional divisor E. Then*

(i) *If E is a $(-1, r)$-cycle, then $x \in X$ is locally analytically isomorphic to*

$$T_{2,3,r+7}\colon \quad 0 \in \{z^2 = (y + x^2)(y^2 + x^{r+5})\} \subset \mathbb{C}^3.$$

(ii) *If E is a $(-2, r)$-cycle, then $x \in X$ is locally analytically isomorphic to*

$$T_{2,4,r+5}\colon \quad 0 \in \{z^2 = x(y + x)(y^2 + x^{r+3})\} \subset \mathbb{C}^3.$$

(iii) *If E is a $(-1, -1, r_1, r_2)$-cycle, then $x \in X$ is locally analytically isomorphic to*

$$T_{2,r_1+5,r_2+5}\colon \quad 0 \in \{z^2 = (y^2 + x^{r_1+3})(x^2 + y^{r_2+3})\} \subset \mathbb{C}^3.$$

Proof. By the classification in [Lau77, Section V], cusp singularities of the above types are determined up to change of co-ordinates by the form of their exceptional divisors. The local forms are taken from [Lau77, Table 1] and [Lau77, Table 2]. □

Finally, the degenerate cusps are the most esoteric. We won't say much about them here, aside from the observation that the *double pinch points* $0 \in \{z^2 = y^2(y+x^2)\} \subset \mathbb{C}^3$ and $0 \in \{z^2 = x^2y^2\} \subset \mathbb{C}^3$ are both degenerate cusps [SB83a, Section 1].

3.2 Degenerations of K3 Surfaces of Degree Two

The aim of this chapter is to explicitly classify the fibres appearing in the relative log canonical models of semistable terminal threefolds fibred by K3 surfaces of degree two. Our reason for doing this is simple. As mentioned in Section 2.1, we would like to emulate Catanese's and Pignatelli's explicit construction of the relative canonical model for a surface fibred by genus two curves. However, their construction relies heavily on a theorem of Mendes-Lopes [ML89, Theorem 3.7] that classifies the canonical rings of degenerate genus two curves. Thus in order to emulate their construction, we need to have an explicit description of the singular fibres occurring in our relative log canonical models. In order to perform this classification, we rely heavily upon results from the birational geometry of degenerations; for an overview of this theory, see Section 2.3.

We begin by recalling the set up from Section 2.4. Let S denote a nonsingular complex curve and let $(X, \pi, \mathcal{L})$ be a semistable terminal threefold fibred by K3 surfaces of degree two over S. After twisting the polarisation by $\mathcal{O}_X(Z)$, for some divisor Z supported on finitely many fibres of π, we may further assume that the polarisation $\mathcal{L}$ is locally π-flat.

With this setup we may then find $\pi'' \colon X'' \to S$ birational to $\pi \colon X \to S$ and a line bundle $\mathcal{L}''$ on X'' such that $(X'', \pi'', \mathcal{L}'')$ is a semistable analytic threefold fibred by K3 surfaces of degree two, X'' is locally Kulikov, $\mathcal{L}''$ is π''-nef and locally π''-flat, and $(X'', \mathcal{L}'')$ has the same relative log canonical model as $(X, \mathcal{L})$. Moreover, by Theorem 2.4.9, the map $\phi'' \colon X'' \to (X'')^c$ of X'' to its relative log canonical model is a morphism. Note, however, that in return for this structure we may lose algebraicity of X''. We have a diagram

$$\begin{tikzcd}
X \arrow[r, dashed, "\phi"] \arrow[d, dashed] & X^c := \mathbf{Proj}_S \mathcal{R}(X, \mathcal{L}) \arrow[d, equal] \\
X'' \arrow[r, "\phi''"] \arrow[dr, "\pi''"'] & (X'')^c := \mathbf{Proj}_S \mathcal{R}(X'', \mathcal{L}'') \arrow[d, "\pi^c"] \\
& S
\end{tikzcd}$$

With this in place, we are ready to start analysing the fibres of the relative log canonical model $\pi^c \colon (X'')^c \to S$. We begin by using the results of Section 2.3 to form a coarse classification of the fibres of $\pi'' \colon X'' \to S$. By definition, the fibre X''_s over s is a general fibre if and only if it is a K3 surface of degree two. In this case, by Example 1.1.4, $\mathcal{L}''_s$ is generated by its global sections and the fibre X''_s is hyperelliptic.

Using Theorem 2.3.5, we may write down a coarse classification of the fibres in X'' where one of these conditions fails, and hence classify the special fibres of X'':

(Ih) X''_s is a smooth K3 surface, $\mathcal{L}''_s$ is generated by its global sections but is not ample.

(Iu) X''_s is a smooth K3 surface, $\mathcal{L}''_s$ is not generated by its global sections.

(II) X''_s is a chain of elliptic ruled components with rational surfaces at each end, and all double curves are smooth elliptic curves.

(III) X''_s consists of rational surfaces meeting along rational curves which form cycles in each component, the dual graph of X''_s is a tiling of the sphere S^2.

We call a general fibre *Type 0* and a special fibre *Type Ih, Iu, II or III*, depending upon which case it satisfies.

In order to study the images of these fibres under the birational map ϕ'', it will be convenient to restrict ourselves to a small neighbourhood of a chosen fibre X''_s. This will make the situation we are considering much simpler and allow us to use many of the techniques developed in Section 2.3 to study the properties of this chosen fibre. First, however, we need to check that we can safely do this.

Suppose that we wish to study the fibre X''_s over $s \in S$. Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and let $i\colon \Delta \to S$ be an embedding with $i(0) = s$. Let $X''(s)$ be defined by the pull-back:

$$\begin{tikzcd} X''(s) \arrow[r, "j"] \arrow[d, "\pi''(s)"'] & X'' \arrow[d, "\pi''"] \\ \Delta \arrow[r, "i"] & S \end{tikzcd}$$

We wish to show that the restriction of $(X'')^c$ to Δ agrees with the relative log canonical model of the pair $(X''(s), j^*\mathcal{L}'')$, given by

$$X''(s)^c := \mathbf{Proj}_{\Delta}\mathcal{R}(X''(s), j^*\mathcal{L}'').$$

By definition, $(X'')^c$ is defined by the direct images $\pi''_*(\omega^n_{X''} \otimes (\mathcal{L}'')^n)$ for $n > 0$, so the restriction of $(X'')^c$ to Δ is defined by the restriction of these direct images to Δ, given by $i^*\pi_*(\omega_{X''} \otimes (\mathcal{L}'')^n)$. On the other hand, since X'' is locally Kulikov, $\omega_{X''(s)} \cong \mathcal{O}_{X''(s)}$

and $X''(s)^c$ is defined by the direct images $\pi''(s)_*(j^*\mathcal{L}'')^n$ for $n > 0$. The fact that these two maps agree on $X''(s)$ follows from the $k = 0$ case of:

Lemma 3.2.1. *For all $n > 0$ and $k \geq 0$, there is a natural isomorphism*

$$i^* R^k \pi''_*(\omega^n_{X''} \otimes (\mathcal{L}'')^n) \cong R^k \pi''(s)_*(j^*\mathcal{L}'')^n.$$

Proof. Firstly note that $j^*\omega_{X''} \cong \omega_{X''(s)} \cong \mathcal{O}_{X''(s)}$, so the right hand side is isomorphic to $R^k\pi''(s)_*((j^*\omega_{X''})^n \otimes (j^*\mathcal{L}'')^n)$. Then, since tensor product commutes with inverse image, this is equal to $R^k\pi''(s)_* j^*(\omega^n_{X''} \otimes (\mathcal{L}'')^n)$. Finally, since taking higher direct images commutes with restriction to open subsets, we have an isomorphism $R^k\pi''(s)_* j^*(\omega^n_{X''} \otimes (\mathcal{L}'')^n) \cong i^* R^k \pi''_*(\omega^n_{X''} \otimes (\mathcal{L}'')^n)$. $\square$

Furthermore, as $\mathcal{L}''$ is π''-nef and locally π''-flat, we may assume $j^*\mathcal{L}'' = \mathcal{O}_{X''(s)}(H)$ for some effective, nef and $\pi''(s)$-flat divisor H on $X''(s)$.

With this in place, we are ready to begin studying the explicit form of the fibres appearing in $(X'')^c$. By Lemma 3.2.1, it is enough to consider this problem locally around any fibre X''_s. After a slight change of notation, we obtain the following classification:

Theorem 3.2.2. *Let $\pi\colon X \to \Delta$ be a semistable degeneration of K3 surfaces, with $\omega_X \cong \mathcal{O}_X$. Let $\mathcal{L} = \mathcal{O}_X(H)$ for some effective, nef and π-flat divisor H on X, and suppose that $\mathcal{L}$ induces a polarisation of degree two on the general fibre. Then:*

If $X_0 = \pi^{-1}(0)$ is a fibre of Type 0, then $\mathcal{L}$ is ample and the morphism $\phi\colon X \to X^c$ taking X to the relative log canonical model of the pair $(X, \mathcal{L})$ is an isomorphism that sends X_0 to:

(0) *A sextic hypersurface*

$$\{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

where $\{f_6(x_i) = 0\} \subset \mathbb{P}^2$ is a nonsingular curve.

If X_0 is one of the special fibres (I)-(III) *then the morphism $\phi\colon X \to X^c$ sends X_0 to:*

(Ih) *A sextic hypersurface*

$$\{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

where $\{f_6(x_i) = 0\} \subset \mathbb{P}^2$ has at worst A-D-E singularities.

(Iu) *A complete intersection*

$$\{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$

where $\{f_2(x_i) = 0\} \subset \mathbb{P}^2$ is nonsingular, $f_6(0,0,0,1) \neq 0$ and the branch curve $\{f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}(1,1,1,2)$ has at worst A-D-E singularities.

(II.0h) *A sextic hypersurface*

$$\{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

where the curve $\{f_6(x_i) = 0\} \subset \mathbb{P}^2$ has a consecutive triple point or a quadruple point, along with some A-D-E singularities.

(II.0u) *A complete intersection*

$$\{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$

where $\{f_2(x_i) = 0\} \subset \mathbb{P}^2$ is nonsingular, $f_6(0,0,0,1) \neq 0$ and the branch curve $\{f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}(1,1,1,2)$ has a consecutive triple point or a quadruple point, along with some A-D-E singularities.

(II.1) *A sextic hypersurface*

$$\{z^2 - l^2(x_i)f_4(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

where $\{l(x_i) = 0\} \subset \mathbb{P}^2$ *is a line that intersects* $\{f_4(x_i) = 0\}$ *in four distinct points and either*

(II.1a) $\{f_4(x_i) = 0\} \subset \mathbb{P}^2$ *has at worst A-D-E singularities; or*

(II.1b) $\{f_4(x_i) = 0\} \subset \mathbb{P}^2$ *has a quadruple point.*

(II.2) *A sextic hypersurface*

$$\{z^2 - q^2(x_i)f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

where $\{q(x_i) = 0\} \subset \mathbb{P}^2$ *is a nonsingular quadric curve that intersects* $\{f_2(x_i) = 0\}$ *in four distinct points and* $\{f_2(x_i) = 0\} \subset \mathbb{P}^2$ *has at worst A-D-E singularities.*

(II.3) *A sextic hypersurface*

$$\{z^2 - f_3^2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

where f_3 *is a nonsingular cubic.*

(II.4) *A complete intersection given, up to linear change of coordinates in the* x_i*, by*

$$\{z^2 - f_6(x_i, y) = x_2x_3 = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$

where the locus $\{f_6(x_1, 0, 0, y) = 0\} \subset \mathbb{P}(1, 2)$ *consists of three distinct points;* $f_6(0, 0, 0, 1) \neq 0$*; and either*

(II.4a) $\{f_6(x_i, y) = x_j = 0\} \subset \mathbb{P}(1, 1, 1, 2)$ *has at worst A-D-E singularities for* $j = 2, 3$*; or*

(II.4b) $\{f_6(x_i, y) = x_j = 0\} \subset \mathbb{P}(1,1,1,2)$ *has a consecutive triple point for exactly one choice of* $j \in \{2,3\}$ *and at worst A-D-E singularities for the other.*

(III.0h) *A sextic hypersurface*

$$\{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

that has exactly one of the cusp singularities listed in Proposition 3.1.9, possibly along with some rational double points.

(III.0u) *A complete intersection*

$$\{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$

where $\{f_2(x_i) = 0\} \subset \mathbb{P}^2$ *is nonsingular,* $f_6(0,0,0,1) \neq 0$ *and this complete intersection has exactly one of the cusp singularities listed in Proposition 3.1.9, along with some rational double points.*

(III.1) *A sextic hypersurface*

$$\{z^2 - l^2(x_i) f_4(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

where $\{f_4(x_i) = 0\} \subset \mathbb{P}^2$ *has at worst A-D-E singularities;* l *is linear; and the curves in* $\mathbb{P}^2$ *defined by* $\{l(x_i) = 0\}$ *and* $\{f_4(x_i) = 0\}$ *intersect in two or three distinct points, each with multiplicity* ≤ 2.

(III.2) *A sextic hypersurface*

$$\{z^2 - q^2(x_i) f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

where $\{f_2(x_i) = 0\} \subset \mathbb{P}^2$ *has at worst A-D-E singularities;* q *is a quadric with at worst nodal singularities; and the curves in* $\mathbb{P}^2$ *defined by* $\{q(x_i) = 0\}$ *and*

$\{f_2(x_i) = 0\}$ *intersect in two, three or four distinct points, each with multiplicity* ≤ 2*. Furthermore, if* $\{q(x_i) = 0\}$ *and* $\{f_2(x_i) = 0\}$ *intersect in four points then* q *must have a nodal singularity.*

(III.3) *A sextic hypersurface*

$$\{z^2 - f_3^2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z],$$

where f_3 *is a singular cubic with nodal singularities.*

(III.4) *A complete intersection given, up to linear change of coordinates in the* x_i*, by*

$$\{z^2 - f_6(x_i, y) = x_2x_3 = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$

where $\{f_6(x_1, 0, 0, y) = 0\} \subset \mathbb{P}(1, 2)$ *consists of exactly two points, one of which has multiplicity 2;* $f_6(0, 0, 0, 1) \neq 0$*; and either*

(III.4a) $\{f_6(x_i, y) = x_j = 0\} \subset \mathbb{P}(1, 1, 1, 2)$ *has at worst A-D-E singularities for* $j = 2, 3$*; or*

(III.4b) $\{f_6(x_i, y) = x_j = 0\} \subset \mathbb{P}(1, 1, 1, 2)$ *has at worst A-D-E singularities for one choice of* $j \in \{2, 3\}$ *and consists of two nonsingular curves, one of which is double, meeting in a degenerate cusp* $0 \in \{y^2(y + x^2) = 0\} \subset \mathbb{C}$ *for the other.*

Remark 3.2.3. We briefly comment on the relationship between this theorem and two other classifications of semistable fibres.

Firstly, in [Fri84, Section 5], Friedman classifies the semistable fibres occurring in a Type II degeneration of K3 surfaces of degree two. However, in his classification he drops the assumption that H is π-flat. This allows him to twist H by components of the central fibre X_0, which enables him to make some strong assumptions about the form of the restriction of H to X_0 (see [Fri84, Theorem 2.2]). This makes his classification considerably simpler than ours: in fact, only cases (II.1a), (II.2), (II.3) and (II.4a) appear. We originally considered taking this approach to our classification but, as we shall see in Chapter 4, the π-flat assumption turns out to be quite important when we come to explicitly construct the relative log canonical model.

Secondly, in [Sha80, Theorem 2.4], Shah classifies the semistable degenerations of sextic curves in $\mathbb{P}^2$. This is closely related to our setup, because any K3 hypersurface in $\mathbb{P}(1,1,1,3)$ is a double cover of $\mathbb{P}^2$ ramified over a nonsingular sextic curve. Thus, a semistable degeneration of K3 hypersurfaces in $\mathbb{P}(1,1,1,3)$ corresponds to a semistable degeneration of sextic curves in $\mathbb{P}^2$. There are two main differences between our classifications. The first is that Shah's classification has far fewer distinct cases. This is because, in the interests of constructing a good moduli space, Shah considers only those sextics belonging to closed orbits, but we do not impose this criterion. The second is that our classification treats the unigonal fibres differently. In Shah's classification the unigonal fibres are seen as double covers of $\mathbb{P}^2$, obtained by applying the natural projection $\mathbb{P}(1,1,1,2,3) - \to \mathbb{P}(1,1,1,3)$. However, as was noted in Example 2.1.1, applying this projection to a unigonal fibre gives a non-reduced fibre, which is difficult to deal with.

Before we embark on the proof of Theorem 3.2.2, we note that the following corollary is an immediate consequence of the classification in the theorem and the considerations in Section 3.1.

Corollary 3.2.4. *The singularities appearing in the singular fibres of the relative log canonical model of a semistable terminal threefold fibred by K3 surfaces of degree two are at worst semi log canonical.*

We now proceed with the proof of Theorem 3.2.2. To do this, we will show that the log canonical model

$$(X_0)^c := \mathrm{Proj} \bigoplus_{n \geq 0} H^0(X_0, \mathcal{L}_0^n)$$

of the pair $(X_0, \mathcal{L}_0)$, where $\mathcal{L}_0$ denotes the sheaf induced on X_0 by $\mathcal{L}$, agrees with the central fibre $(X^c)_0$ of $\pi^c \colon X^c \to \Delta$. Furthermore we will show that, after some elementary modifications (which do not affect the form of the relative log canonical model, by Lemma 2.3.7) have been performed on X, the image of the natural morphism $\phi_0 \colon X_0 \to (X_0)^c$ can be explicitly calculated. As ϕ_0 agrees with the restriction of ϕ to X_0, this will be enough to prove the theorem.

Our first step is to show that $(X^c)_0$ and $(X_0)^c$ agree. As noted above, the log canonical model of the pair $(X_0, \mathcal{L}_0)$ is defined by the global sections $H^0(X_0, \mathcal{L}_0^n)$ for $n > 0$. On the other hand, the central fibre $(X^c)_0$ of $\pi^c \colon X^c \to \Delta$ is defined by the localised direct images $\pi_*(\mathcal{L}^n)_0 \otimes_{\mathcal{O}_{\Delta,0}} k(0)$ for $n > 0$, where $k(0)$ is the residue field at $0 \in \Delta$. The fact that these two maps agree will follow from:

Lemma 3.2.5. *For all $n > 0$ and all $i > 0$, suppose that $R^i\pi_*(\mathcal{L}^n) = 0$ and $\pi_*(\mathcal{L}^n)$ is locally free. Then the natural maps*

$$\pi_*(\mathcal{L}^n)_0 \otimes_{\mathcal{O}_{\Delta,0}} k(0) \longrightarrow H^0(X_0, \mathcal{L}_0^n)$$

are isomorphisms for all $n > 0$.

Proof. By [BS76, Corollary III.3.7] and [BS76, Corollary III.3.10], the conditions of the lemma imply that the natural maps

$$\pi_*(\mathcal{L}^n)_0 \longrightarrow H^0(X_0, \mathcal{L}_0^n)$$

are surjective. But, by [BS76, Theorem III.3.4], this is equivalent to the natural maps

$$\pi_*(\mathcal{L}^n)_0 \otimes_{\mathcal{O}_{\Delta,0}} k(0) \longrightarrow H^0(X_0, \mathcal{L}_0^n),$$

being isomorphisms, as required. ∎

Note that the assumption that $R^i\pi_*(\mathcal{L}^n) = 0$ for all $i > 0$ and all $n > 0$ follows from Corollary 2.4.10 and Lemma 3.2.1, and the assumption that $\pi_*(\mathcal{L}^n)$ is locally free for all $n > 0$ follows from Lemma 1.3.4 and Lemma 3.2.1, so the consequences of the lemma hold and $(X^c)_0$ and $(X_0)^c$ agree.

We are now ready to begin analysing the different cases. As the full proof is rather long and the cases are not covered in order, for ease of cross-referencing the proof of each case will be marked by the number of that case in bold face. We begin with the easiest cases: those of fibres of Types 0, Ih and Iu.

(3.2.2) Case 0. We begin by considering fibres of Type 0. In this case $\mathcal{L}$ is π-ample, so the morphism $\phi\colon X \to X^c$ is an isomorphism. It just remains to calculate an explicit form for the image of X_0 under ϕ.

By Lemma 3.2.5, the restriction of ϕ to X_0 agrees with ϕ_0. So, since $\mathcal{L}_0$ is ample, by Example 1.1.4, ϕ_0 is an isomorphism onto a nonsingular sextic hypersurface in $\mathbb{P}(1,1,1,3)$.

Next, let $\pi\colon X \to \Delta$ be a degeneration of Type Ih or Iu, and let $\mathcal{L}$ be defined as in the statement of Theorem 3.2.2. Then Lemma 3.2.5 shows that the restriction of $\mathcal{L}$ to the central fibre is well behaved, so we can continue with the analysis of the cases.

(3.2.2) Case Ih. We consider fibres of Type Ih first. Here X_0 is a nonsingular K3 surface and the restriction $\mathcal{L}_0$ of $\mathcal{L}$ to X_0 is generated by its global sections but is not ample. However, by construction it must be nef. Furthermore $\mathcal{L}_0.\mathcal{L}_0 = 2$ so, by [MM07, Corollary 2.3.38], $\mathcal{L}_0$ is also big. Hence, by Example 1.1.5, ϕ_0 is a morphism onto a sextic hypersurface $\{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z]$ that has at worst rational double point singularities. Such singularities can only arise from A-D-E singularities in the branch curve $f_6(x_i) = 0$.

(3.2.2) Case Iu. It makes sense to consider fibres of Type Iu next, as these are quite similar to the Type Ih fibres. Here X_0 is a nonsingular K3 surface and $\mathcal{L}_0$ is not generated by its global sections. In this case, by Example 1.1.5, ϕ_0 maps X_0 to a complete intersection $\{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z]$, where $f_6(0,0,0,1) \neq 0$. The only curves contracted by ϕ_0 are (-2)-curves, which result in at worst rational double point singularities. These singularities can only arise from A-D-E singularities in the branch curve $\{f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}(1,1,1,2)$ and the vertex of the quasismooth cone $\{f_2(x_i) = 0\} \subset \mathbb{P}(1,1,1,2)$.

The Type II and Type III fibres are considerably more complicated than those already covered, so will be discussed in their own sections. Before we can do this, however, we will need to collect some results on the components of such fibres.

3.3 Components of Degenerate Fibres

By the classification of Kulikov models (Theorem 2.3.5), the central fibre in a Type II or III degeneration of K3 surfaces is a union of rational and elliptic ruled components meeting transversely along a set of double curves. In this section we will study the interaction of the polarisation bundle $\mathcal{L}$ with these components.

In order to do this, we begin by fixing some notation. Let $\pi \colon X \to \Delta$ be a degeneration of K3 surfaces with $X_0 := \pi^{-1}(0)$ a fibre of Type II or III. Write $X_0 = V_0 \cup \cdots \cup V_r$ where the V_i are the irreducible components of X_0 and we assume that the V_i have been normalised. Let D_{ij} denote the double curve $V_i \cap V_j$ and let $D_i = \bigcup_j D_{ij}$ denote the double locus on V_i. Let $\mathcal{L}$ denote a line bundle that induces a polarisation of degree two on a general fibre and let H denote an effective, nef and π-flat divisor in the linear system defined by $\mathcal{L}$. Finally, let H_i denote the effective (or zero) divisor obtained by intersecting the divisor H with the component V_i.

To study the behaviour of the polarisation on V_i, we follow Shepherd-Barron [SB83b] and start by separating the V_i into three sets according to the properties of H_i. We will call a curve $C \subset V_i$ a 0-*curve* if $H_i.C = 0$. Then V_i will be called a 0-*surface* if it contains only finitely many 0-curves; a 2-*surface* if H_i is numerically trivial; and a 1-*surface* if it contains a pencil of 0-curves but is not a 2-surface. Note that these classes are mutually exclusive and that, by [SB83b, Proposition 2.3], every component of X_0 is either a 0-, 1- or 2-surface.

This classification will be useful because, as we shall see later, the map ϕ to the relative log canonical model of the pair $(X, \mathcal{L})$ defines a birational morphism on each 0-surface, contracts each 1-surface to a curve and contracts each 2-surface to a point. This observation will allow us to calculate the possible images of X_0 under ϕ by studying the possible configurations of 0-, 1- and 2-surfaces that can occur in it.

The following result about 0-, 1- and 2-surfaces follows easily from [SB83b, Proposition 2.3]:

Proposition 3.3.1. *Suppose that V_i and H_i are defined as above. Then*

(i) *If V_i is a 1-surface, then the pencil of 0-curves on V_i forms a ruling.*

(ii) *If V_i contains a pencil of 0-curves and another 0-curve that does not lie in this pencil, then V_i is a 2-surface.*

(iii) *If V_i contains an effective divisor E that has $E^2 > 0$ and $H_i.E = 0$, then V_i is a 2-surface.*

Proof. (i) and (ii) are immediate from [SB83b, Proposition 2.3]. (iii) follows easily from the Hodge Index Theorem. □

In addition to this, we have the following information about 0-surfaces:

Lemma 3.3.2. *V_i is a 0-surface if and only if $H_i^2 > 0$. Furthermore, this implies that any 0-surface is projective.*

Proof. First assume that V_i is a 0-surface. Then $H_i^2 > 0$ by [SB83b, Lemma 2.8].

Next, assume that $H_i^2 > 0$. Then H_i cannot be numerically trivial, so V_i is not a 2-surface. So suppose that V_i is a 1-surface. Then, by Proposition 3.3.1, V_i is ruled and the pencil of 0-curves form a ruling. Let F be any such 0-curve. Then $F^2 = 0$ and, since $H_i^2 > 0$, the Hodge Index Theorem implies that F is numerically equivalent to 0. But F is a fibre of a ruling, so this cannot occur. Thus, V_i is not a 1- or 2-surface so, by [SB83b, Proposition 2.3], it must be a 0-surface.

Finally, to show that V_i is projective, note that H_i is nef and, since V_i is Kähler and $H_i^2 > 0$, by [MM07, Corollary 2.3.38] it is also big. Hence, by [MM07, Theorem 2.2.15], V_i is a Moishezon manifold and so, by [MM07, Theorem 2.2.26], V_i is projective. □

For the rest of this section, we must separate the cases where the components under consideration are rational or elliptic ruled. This distinction will enable us to get much more information about the components themselves and the polarisation divisors on them.

3.3.1 Rational Components

In this subsection we will perform a brief study of the rational components occurring in the central fibre of a Kulikov model of a degeneration of K3 surfaces. Note that such components can appear in either Type II or Type III degenerations, and as such are much more general than the elliptic ruled components studied later. To study these components, we begin with a brief digression on the theory of *anticanonical pairs*.

Definition 3.3.3 [Fri83, 1]. *An* anticanonical pair (V, D) *consists of a rational surface* V *and a reduced section* $D \in |-K_V|$ *(which is necessarily connected).*

Friedman [Fri83] proves several results on anticanonical pairs that will be very useful to us. We state these below:

Lemma 3.3.4 [Fri83, Lemma 3]. *If* (V, D) *is an anticanonical pair and* C *is an irreducible curve on* V *which is not a component of* D*, then:*

(1) *If* $C^2 < 0$*, then* C *is smooth rational and either*

(1.1) $C^2 = -2$, $C.D = 0$ *or*

(1.2) $C^2 = -1$, $C.D = 1$.

(2) *If* $C^2 = 0$, *then either*

(2.1) C *is smooth rational,* $C.D = 2$ *or*

(2.2) $p_a(C) = 1$ *and* $C.D = 0$.

Theorem 3.3.5 [Fri83, Theorem 10]. *Let H be effective and nef on V, with (V, D) an anticanonical pair, and suppose that no component of D is a fixed component of $|H|$. Write $|H| = |H|_f + |H|_m$, where $|H|_f$ is the fixed part of $|H|$ and $|H|_m$ has no fixed components. Then:*

(1) $|H|_f \neq 0$ *only if* $H^2 > 0$*; in this case* $|H|_f$ *is either 0 or*

(1.1) $|H|_f = C$ *smooth rational,* $C^2 = -2$, *with* $|H|_m = kE$ *for smooth elliptic* E *with* $E^2 = E.D = 0$ *and* $C.E = 1$. *Necessarily* $k \geq 2$.

(1.2) $|H|_f = \sum_{i=1}^{n} C_i$, *where* $C_i^2 = -2$ *for* $1 \leq i < n$, $C_n^2 = -1$, *with* $C_i.C_{i+1} = 1$, $C_i.C_j = 0$ *for* $j \neq i \pm 1$ *or* i. *In this case,* $|H|_m$ *either contains smooth irreducible members and has* $|H|_m.C_1 = 1$ *or is of the form* kE *as in* (1.1) *and has* $|H|_m.C_1 = k$. *Furthermore,* $|H|_m.C_i = 0$ *for all* $i > 1$.

(2) *If* $|H|$ *has no fixed components,* $|H|$ *has base points if and only if* $H.D = 1$, *in which case* $p = H.D$ *is the unique base point of* $|H|$ *and* $|H|$ *contains smooth members.*

(3) *If* $H^2 = 0$ *then either*

(3.1) $H = kC$ *for* C *smooth rational with* $C^2 = 0$, $C.D = 2$, *and the morphism to projective space defined by* $|C|$ *maps onto* $\mathbb{P}^1$, *exhibiting* V *as a ruled surface.*

(3.2) $H = kE$, *where* E *is as in* (1.1).

(4) *If $H^2 > 0$, then $|nH|$ has no fixed components or base locus for $n \geq 2$. If $n \geq 3$, the morphism to projective space defined by $|nH|$ is birational onto its image and contracts exactly those curves C with $H.C = 0$.*

The reason that this is useful to us becomes clear from the following lemma:

Lemma 3.3.6. *Let V_i be a rational component of the central fibre X_0 in a degeneration of K3 surfaces $\pi \colon X \to \Delta$ of Type II or III, and let D_i be the locus of double curves on V_i. Then (V_i, D_i) is an anticanonical pair.*

Proof. As D_i is reduced, we just need to show that $D_i \in |-K_{V_i}|$. This is stated without proof in [Per77], we provide a proof here for completeness. Since V_i is a component in the central fibre X_0 of a Kulikov model $\pi \colon X \to \Delta$, by adjunction we get that $K_{V_i} \sim (K_X + V_i)|_{V_i} \sim V_i.V_i$. Next, note that since $X_0 \sim X_s$ for X_s a generic fibre, $X_0.V_i \sim 0$. So $V_i.V_i \sim V_i.(V_i - X_0) \sim V_i.(-\sum_{i \neq j} V_j)$, where V_j runs over the other components of X_0. But this intersection is precisely $-D_i$. $\square$

We conclude this subsection with a result that will allow us to calculate the ranks of the cohomology groups of a divisor on an anticanonical pair. This will prove invaluable when we want to calculate the image of such a pair under the map to its log canonical model.

Proposition 3.3.7. *Let (V, D) be an anticanonical pair. Then:*

(a) *If C is a smooth irreducible curve on V satisfying $C^2 \geq 0$ and $C.D > 0$,*

$$h^0(V, \mathcal{O}_V(nC)) = \frac{1}{2}C^2 n^2 + \left(\frac{1}{2}C^2 - p_a(C) + 1\right) n + 1.$$

(b) *If E is an effective, nef divisor on V with $E^2 > 0$ and $E \cap D = \emptyset$,*

$$h^0(V, \mathcal{O}_V(nE)) = \frac{1}{2}E^2 n^2 + 2.$$

Proof. We begin by proving (a). Consider the exact sequence for $n \geq 1$:

$$0 \longrightarrow \mathcal{O}_V((n-1)C) \longrightarrow \mathcal{O}_V(nC) \longrightarrow \mathcal{O}_C(nC) \longrightarrow 0$$

This gives rise to the long exact sequence of cohomology

$$\begin{aligned} 0 &\longrightarrow H^0(V, \mathcal{O}_V((n-1)C)) \longrightarrow H^0(V, \mathcal{O}_V(nC)) \longrightarrow H^0(C, \mathcal{O}_C(nC)) \\ &\longrightarrow H^1(V, \mathcal{O}_V((n-1)C)) \longrightarrow H^1(V, \mathcal{O}_V(nC)) \longrightarrow H^1(C, \mathcal{O}_C(nC)) \\ &\longrightarrow \cdots . \end{aligned}$$

We will use this sequence and induction on n to prove the result we want. Firstly note that, by adjunction and Serre duality,

$$H^1(C, \mathcal{O}_C(nC)) \cong H^0(C, \mathcal{O}_C(-D-(n-1)C)).$$

But this second group is zero, as $\deg_C(-D-(n-1)C) < 0$. Secondly, $H^1(V, \mathcal{O}_V) = 0$ as V is rational. So, by induction, $H^1(V, \mathcal{O}_V(nC)) = 0$ for all n.

Hence, from the long exact sequence above, we get that

$$h^0(V, \mathcal{O}_V(nC)) = h^0(V, \mathcal{O}_V((n-1)C)) + h^0(C, \mathcal{O}_C(nC)).$$

Noting that $h^0(V, \mathcal{O}_V) = 1$, we will be done by induction on n if we can show that $h^0(C, \mathcal{O}_C(nC)) = nC^2 + 1 - p_a(C)$. But this follows immediately from the Riemann-Roch theorem for curves and the vanishing of $H^1(C, \mathcal{O}_C(nC))$.

Next we prove (b). Note that V is a 0-surface with respect to E, so it must be projective by Lemma 3.3.2. Thus we may use the Riemann-Roch Theorem for surfaces to calculate $h^0(V, \mathcal{O}_V(nE))$ for $n \geq 0$.

We begin by calculating $h^i(V, \mathcal{O}_V(nE))$ for $i = 1, 2$. By Serre duality, we have

$$H^i(V, \mathcal{O}_V(nE)) \cong H^{2-i}(V, \mathcal{O}_V(-nE - D)).$$

We will calculate this second group by means of the exact sequence

$$0 \longrightarrow \mathcal{O}_V(-nE - D) \longrightarrow \mathcal{O}_V(-nE) \longrightarrow \mathcal{O}_D(-nE) \longrightarrow 0$$

which gives rise to the long exact sequence of cohomology

$$\begin{aligned} 0 \longrightarrow H^0(V, \mathcal{O}_V(-nE - D)) \longrightarrow H^0(V, \mathcal{O}_V(-nE)) \longrightarrow H^0(D, \mathcal{O}_D(-nE)) \\ \longrightarrow H^1(V, \mathcal{O}_V(-nE - D)) \longrightarrow H^1(V, \mathcal{O}_V(-nE)) \longrightarrow \cdots . \end{aligned}$$

Now, E is nef by assumption. Furthermore, by [KM98, Proposition 2.61], the assumption $E^2 > 0$ implies that E is big. So, by the generalised Kodaira vanishing theorem [KM98, Theorem 2.70],

$$H^i(V, \mathcal{O}_V(-nE)) = 0$$

for all $i < 2$. This immediately gives $H^0(V, \mathcal{O}_V(-nE - D)) = 0$ and

$$H^1(V, \mathcal{O}_V(-nE - D)) \cong H^0(D, \mathcal{O}_D(-nE)).$$

However, as $E \cap D = \emptyset$, we have $\mathcal{O}_D(-nE) \cong \mathcal{O}_D$ and so, as D is connected,

$$h^0(D, \mathcal{O}_D(-nE)) = h^0(D, \mathcal{O}_D) = 1.$$

Putting this into the Riemann-Roch Theorem for surfaces, noting that $\chi(\mathcal{O}_V) = 1$

(as V is rational), we get

$$h^0(V, \mathcal{O}_V(nE)) - 1 = 1 + \frac{1}{2}E^2n^2.$$

Rearranging gives the desired result. □

This concludes the analysis of the rational components.

3.3.2 Elliptic Ruled Components

Finally, we will perform a brief study of the elliptic ruled components occurring in the central fibre of a Kulikov model of a degeneration of K3 surfaces. Our aim is to study the structure of these components and to prove analogues of some of the results on rational surfaces quoted in Subsection 3.3.1.

By the classification of Kulikov models (Theorem 2.3.5), elliptic ruled components can only appear in a degeneration of Type II. Furthermore, each elliptic ruled component in a Type II degeneration contains precisely two smooth elliptic double curves that form sections for the ruling.

We begin by studying the general form of a minimal elliptic ruled surface.

Lemma 3.3.8. *Let V be a minimal analytic surface, C a smooth curve, and $p\colon V \to C$ a morphism with generic fibre isomorphic to $\mathbb{P}^1$. Then V is C-isomorphic to a projective space bundle $\mathbb{P}_C(\mathcal{E})$, where $\mathcal{E}$ is some algebraic rank 2 vector bundle on C. In particular, this implies that V is projective.*

Proof. This is a special case of [BHPvdV04, Proposition V.4.1]. □

This description of a minimal elliptic ruled surface can be used to get a description of the Picard group of such a surface.

Lemma 3.3.9 [Bea96, Proposition III.18]. *Let $V = \mathbb{P}_C(\mathcal{E})$ be a minimal ruled surface over C and let $p\colon V \to C$ be the structural morphism. Write s for the class of the sheaf $\mathcal{O}_V(1)$ in $\mathrm{Pic}(V)$ (or in $H^2(V,\mathbb{Z})$). Then*

(i) $\mathrm{Pic}(V) = p^*\mathrm{Pic}(C) \oplus \mathbb{Z}s$.

(ii) $H^2(V,\mathbb{Z}) = \mathbb{Z}s \oplus \mathbb{Z}f$, *where f is the class of a fibre.*

(iii) $s^2 = \deg(\mathcal{E})$, $f^2 = 0$ *and* $s.f = 1$.

(iv) $[K_V] = -2s + (\deg(\mathcal{E}) + 2g(C) - 2)f$ *in* $H^2(V,\mathbb{Z})$.

For the remainder of this subsection, V_i will denote an elliptic ruled surface that is a component in the central fibre of a degeneration $\pi\colon X \to \Delta$ of Type II and D_i will denote the double locus on V_i. Write D_i as $D_i' + D_i''$, for D_i', D_i'' smooth irreducible elliptic sections. We will further assume that V_i is minimal (i.e. V_i contains no (-1)-curves).

Using the description of Lemma 3.3.8, without loss of generality we can take $C \cong D_i'$ and $p\colon V \to D_i'$ to be the natural projection. Then, by the proof of [Har77, Proposition V.2.2], $\mathcal{O}_{V_i}(D_i') \cong \mathcal{O}_{V_i}(1)$ and so, using the notation of Lemma 3.3.9, $[D_i'] = s$ in $H^2(V_i,\mathbb{Z})$. We have:

Lemma 3.3.10. *With notation as above, $K_{V_i} \sim -D_i$, so $[D_i''] = s - \deg(\mathcal{E})f$ in $H^2(V,\mathbb{Z})$ and $(D_i'')^2 = -(D_i')^2$.*

Proof. The statement about K_{V_i} is proved in exactly the same way as Lemma 3.3.6. Furthermore, the statements about D_i'' follow immediately from Lemma 3.3.9 and the assumption that $[D_i'] = s$. □

Next, we will study the interaction between the elliptic ruled components and the polarisation divisor. For this we will use the mechanics of 0-, 1- and 2-surfaces developed before. So suppose that H is an effective, nef and flat divisor on X that induces a polarisation of degree two on the general fibre of $\pi\colon X \to \Delta$. Let H_i denote the intersection of H with V_i. Then we have:

Lemma 3.3.11. *With notation as above:*

(i) *If V_i is a 1-surface, then H_i is a sum of fibres and $H_i.D_i' = H_i.D_i''$.*

(ii) *if V_i is a 0-surface, then $[H_i] = as + bf$ in $H^2(V_i, \mathbb{Z})$, for $a > 0$ and $b \geq 0$.*

Proof. Consider (i) first. By Proposition 3.3.1, we know that the pencil of 0-curves on V_i forms a ruling. Let C be a general member of this linear system. Then, by Lemma 3.3.9, $[C] = f$ in $H^2(V_i, \mathbb{Z})$. Write $[H_i] = as + bf$ in $H^2(V_i, \mathbb{Z})$, for some $a, b \in \mathbb{Z}$. Then $0 = H_i.C = (as+bf).f = a$ which implies that $[H_i] = bf$, i.e. that H_i is a sum of fibres. To complete the proof of (i), note that since $[H_i] = bf$, we have $H_i.D_i' = b = H_i.D_i''$.

Now consider (ii). As before, write $[H_i] = as + bf$ for some $a, b \in \mathbb{Z}$. Consider first

$$\begin{aligned} H_i.D_i'' &= (as + bf).(s - \deg(\mathcal{E})f) \\ &= \deg(\mathcal{E})a + b - \deg(\mathcal{E})a \\ &= b. \end{aligned}$$

As D_i'' is irreducible and H_i is nef, this implies that $b \geq 0$.

Next, let F denote a fibre of the ruling on V_i. Then $H_i.F = a$ so, as F is irreducible and H_i is nef, $a \geq 0$. Finally, note that $H_i^2 = a^2\deg(\mathcal{E}) + 2ab > 0$ by Lemma 3.3.2, so $a \neq 0$. □

Finally, we conclude this subsection with a pair of results that will prove invaluable when calculating the maps induced by certain linear systems on a minimal elliptic ruled component.

Proposition 3.3.12. *Suppose that V_i is a minimal elliptic ruled component in a degeneration of Type II. Let D_i' and D_i'' denote the elliptic double curves on V_i. Suppose that*

$(D_i')^2 > 0$. Then

$$\begin{aligned} h^0(V_i, \mathcal{O}_{V_i}(nD_i')) &= \frac{1}{2}(D_i')^2(n^2+n)+1 \\ h^1(V_i, \mathcal{O}_{V_i}(nD_i')) &= 1 \\ h^2(V_i, \mathcal{O}_{V_i}(nD_i')) &= 0 \end{aligned}$$

for all $n \geq 0$.

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{V_i}(nD_i' - D_i'') \longrightarrow \mathcal{O}_{V_i}(nD_i') \longrightarrow \mathcal{O}_{D_i''}(nD_i') \longrightarrow 0.$$

From this, we get the long exact sequence of cohomology

$$\begin{aligned} \cdots \longrightarrow H^1(V_i, \mathcal{O}_{V_i}(nD_i' - D_i'')) \longrightarrow H^1(V_i, \mathcal{O}_{V_i}(nD_i')) \longrightarrow H^1(D_i'', \mathcal{O}_{D_i''}(nD_i')) \\ \longrightarrow H^2(V_i, \mathcal{O}_{V_i}(nD_i' - D_i'')) \longrightarrow H^2(V_i, \mathcal{O}_{V_i}(nD_i')) \longrightarrow 0. \end{aligned}$$

Now, by Lemma 3.3.10, $K_{V_i} \sim -D_i' - D_i''$, so by Serre duality

$$H^j(V_i, \mathcal{O}_{V_i}(nD_i' - D_i'')) = H^{2-j}(V_i, \mathcal{O}_{V_i}(-(n+1)D_i')).$$

Next note that, as D_i' is irreducible and has $(D_i')^2 > 0$, we have $D_i'.E \geq 0$ for all effective divisors E. So D_i' is nef. Furthermore, by [KM98, Proposition 2.61], D_i' is also big. Hence, by the generalised Kodaira vanishing theorem [KM98, Theorem 2.70], we have that

$$H^{2-j}(V_i, \mathcal{O}_{V_i}(-(n+1)D_i')) = 0$$

for all $n \geq 0$ and $j > 0$. This immediately implies that $H^2(V_i, \mathcal{O}_{V_i}(nD_i')) = 0$ for $n \geq 0$

and

$$H^1(V_i, \mathcal{O}_{V_i}(nD_i')) \cong H^1(D_i'', \mathcal{O}_{D_i''}(nD_i')).$$

Furthermore, as $D_i' \cap D_i'' = \emptyset$, we have that $\mathcal{O}_{D_i''}(nD_i') \cong \mathcal{O}_{D_i''}$. So

$$h^1(V_i, \mathcal{O}_{V_i}(nD_i')) = h^1(D_i'', \mathcal{O}_{D_i''}) = 1$$

for $n \geq 0$.

It just remains to calculate $h^0(V_i, \mathcal{O}_{V_i}(nD_i'))$. By the calculation above and the Riemann-Roch theorem for surfaces we have

$$h^0(V_i, \mathcal{O}_{V_i}(nD_i')) - 1 = \chi(\mathcal{O}_{V_i}) + \frac{1}{2}\big((nD_i')^2 - nD_i'.K_{V_i}\big).$$

By Lemma 3.3.10, we know that $nD_i'.K_{V_i} = -n(D_i')^2$. Furthermore, from the properties of minimal ruled surfaces [Bea96, Proposition III.21], we know that $\chi(\mathcal{O}_{V_i}) = 0$. Rearrangement then proves the required formula for $h^0(V_i, \mathcal{O}_{V_i}(nD_i'))$. □

Corollary 3.3.13. *Under the assumptions of Proposition 3.3.12, the linear system $|nD_i'|$ has no fixed components for $n \geq 1$ and no base points for $n \geq 2$. If $n \geq 3$, the morphism $\phi_{nD_i'}$ corresponding to the linear system $|nD_i'|$ is birational onto its image.*

Proof. (Based upon the proof of [Fri83, Theorem 10]) To see that $|D_i'|$ has no fixed components, note that Proposition 3.3.12 implies that the dimension of the linear system $|D_i'|$ is at least 1. So we may choose an effective divisor $Z \in |D_i'|$ that is not equal to D_i'. But, as D_i' is an irreducible curve, Z and D_i' cannot have any components in common.

Now consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{V_i}((n-1)D_i') \longrightarrow \mathcal{O}_{V_i}(nD_i') \longrightarrow \mathcal{O}_{D_i'}(nD_i') \longrightarrow 0.$$

This gives rise to the long exact sequence of cohomology

$$\begin{aligned}0 &\longrightarrow H^0(V_i, \mathcal{O}_{V_i}((n-1)D'_i) \longrightarrow H^0(V_i, \mathcal{O}_{V_i}(nD'_i)) \longrightarrow H^0(D'_i, \mathcal{O}_{D'_i}(nD'_i)) \\ &\longrightarrow H^1(V_i, \mathcal{O}_{V_i}((n-1)D'_i) \longrightarrow H^1(V_i, \mathcal{O}_{V_i}(nD'_i)) \longrightarrow H^1(D'_i, \mathcal{O}_{D'_i}(nD'_i)) \\ &\longrightarrow \cdots .\end{aligned}$$

Furthermore, since $\deg_{D'_i}(nD'_i) = n(D'_i)^2 > 0$ and $K_{D'_i} \sim 0$, Kodaira vanishing gives $H^1(D'_i, \mathcal{O}_{D'_i}(nD'_i)) = 0$.

Thus, the map $H^1(V_i, \mathcal{O}_{V_i}((n-1)D'_i) \to H^1(V_i, \mathcal{O}_{V_i}(nD'_i))$ is surjective. But, by Proposition 3.3.12, both of these groups have rank 1 for $n \geq 2$. So this map must be an isomorphism and the map $H^0(V_i, \mathcal{O}_{V_i}(nD'_i)) \to H^0(D'_i, \mathcal{O}_{D'_i}(nD'_i))$ must be surjective.

Therefore the restriction of $|nD'_i|$ to D'_i defines a complete linear system and, as $\deg_{D'_i}(nD'_i) \geq 2$ for $n \geq 2$, [Har77, Corollary IV.3.2] implies that this complete linear system is base point free. So $|nD'_i|$ must be base point free for $n \geq 2$ as well.

Finally, as $|nD'_i|$ has no fixed components or base points for $n \geq 2$, by Bertini's Theorem we see that a general member $Z \in |nD'_i|$ is smooth. Furthermore, the adjunction formula gives $\mathcal{O}_Z(nD'_i) = \mathcal{O}_Z(K_Z + D'_i + D''_i)$ and, if $n \geq 3$, we must have $\deg_Z(K_Z + D'_i + D''_i) \geq 2g(Z) + 1$. So, by [Har77, Corollary IV.3.2], $\mathcal{O}_Z(nD'_i)$ is very ample on Z for $n \geq 3$, so $\phi_{nD'_i}$ is birational onto its image. $\square$

This completes the analysis of the elliptic ruled components.

3.4 Type II Fibres

In this section we shall classify the Type II fibres. We begin by setting up some notation. Let $\pi \colon X \to \Delta$ be a degeneration of K3 surfaces with $X_0 = \pi^{-1}(0)$ a fibre of Type II. Write $X_0 = V_0 \cup \cdots \cup V_r$, with $V_1, \ldots, V_{r-1}$ elliptic ruled and V_0 and V_r rational. Let D_i denote the elliptic double curve $V_{i-1} \cap V_i$. As usual, $\mathcal{L}$ denotes a line bundle that induces a polarisation of degree two on a general fibre and H denotes an effective, nef and π-flat divisor in the linear system defined by $\mathcal{L}$.

We begin by making elementary modifications to get X and H into a special form that will make it easier for us to perform calculations with them:

Theorem 3.4.1. *Let $\pi\colon X \to \Delta$ be a Type II degeneration of K3 surfaces and H be an effective, nef and π-flat divisor on X which induces an polarisation of degree two on the generic fibre. Then after suitable elementary modifications have been performed on X, we have*

(i) *H is transformed into an effective, nef and π-flat divisor;*

(ii) *for each $1 \leq i \leq r-1$, the surface V_i is minimal ruled;*

(iii) *let H_i denote the effective (or zero) divisor on V_i obtained by intersecting with H. Then no component of the double locus on V_i is fixed in the linear system $|H_i|$.*

Proof. We begin by proving part (ii). In order to do this we apply a modification of Shepherd-Barron's proof of [SB83b, Theorem 1], due originally to Friedman [Fri84, Theorem 2.2]. If V_1 is elliptic ruled and not minimal, there are (†)-curves C meeting D_2 which may be flipped to V_2 by a series of Type I modifications. Repeat this until V_1 is minimal. Similarly, if V_2 is elliptic ruled and not minimal, we may flip all (†)-curves to V_3, and so on. Eventually, we obtain a birational model X' of X with $V_1', \ldots, V_{r-1}'$ minimal ruled. This proves part (ii).

Next we prove part (i). Let H' denote the strict transform of H on X'. Then, as all elementary modifications occur in codimension 2, H' is still effective and flat over Δ. To prove (i) we need to show that H' can be made nef. In order to do this, we apply the following modification of the algorithm of [SB83b, Theorem 1] to H':

(1) Perform Type 0 modifications along all (§)-curves C lying in V_0' or V_r' such that $H'.C < 0$.

(2) Given a (†)-curve C' on V_0' or V_r' such that $H'.C' < 0$, make a series of Type I modifications to flip C' to the opposite end, i.e. to either V_r' or V_0', then repeat step (1).

The proof of [SB83b, Theorem 1] shows that the algorithm above terminates. The result is a birational model X'' of X and a divisor H'' on X'' that satisfy (i) and (ii) in the statement of the theorem.

The proof of (iii) is somewhat more difficult. We proceed by contradiction. Let H_i be the divisor on V_i obtained by intersecting with H and suppose that D_j is a double curve on V_i (so $j \in \{i, i+1\}$) that is fixed in $|H_i|$. We aim to show that this implies that $(D_j|_{V_i})^2 = 0$ and $H_i.D_j = 0$. For then, in the notation of Section 3.3, $D_j|_{V_i}$ is a nonsingular elliptic 0-curve with self-intersection number 0, which cannot exist by [SB83b, Lemma 2.2].

So, in order to prove (iii), we need to show that $(D_j|_{V_i})^2 = 0$ and $H_i.D_j = 0$. The first of these will follow from the triple point formula if we can show that D_j has non-positive self-intersection on both of the components in which it lies. The first step to proving this is to show that D_j is fixed in both of the components in which it lies.

Remark 3.4.2. We note that the next few results are proved in considerably more generality than we need in order to prove Theorem 3.4.1. However, the same results will also be used when we come to analyse the Type III fibres in the next section, and we will need the greater generality there.

Lemma 3.4.3. *Let V_i and V_j be two distinct surfaces meeting along a double curve D_{ij}. Let $\mathcal{L}$ be an invertible sheaf on $V = V_i \cup V_j$, such that there exist sections of $H^0(V, \mathcal{L})$ which are nonvanishing on V_k for each k. Let H_k denote an effective divisor on V_k defined by a nonvanishing section of $H^0(V, \mathcal{L})$. Suppose that D_{ij} is a fixed component of $|H_i|$ on V_i. Then D_{ij} is also a fixed component of $|H_j|$ on V_j.*

Proof. For a contradiction, suppose that D_{ij} is fixed in $|H_i|$ but not in $|H_j|$. Then there exists a section $s \in H^0(V, \mathcal{L})$ such that s restricted to V_j does not vanish on D_{ij}. But then s restricted to V_i defines a divisor linearly equivalent to H_i that does not vanish on D_{ij}. However, this contradicts D_{ij} being a fixed component of $|H_i|$. □

Applying this lemma with $\mathcal{L}$ equal to the restriction of $\mathcal{O}_X(H)$ to X_0, and noting that this restriction defines the complete linear system $|H_i|$ on each component V_i of X_0 by Lemma 3.2.5, we see that D_j is fixed in both of the components in which it lies. The fact that it has non-positive self-intersection on both of these components will follow from another lemma:

Lemma 3.4.4. *Let V_i be a normalised component of the central fibre X_0 in a degeneration of K3 surfaces $\pi\colon X \to \Delta$ of Type II or III, and let D_i be the locus of double curves on V_i. Let $|H_i|$ be a linear system on V_i which contains in its fixed locus an irreducible component D_{ij} of D_i. Then $D_{ij}^2 \leq 0$, and this inequality is strict if $\pi\colon X \to \Delta$ is of Type III and D_{ij} is smooth.*

Proof. Suppose for a contradiction that $D_{ij}^2 > 0$ (or that $D_{ij}^2 \geq 0$ in the case where $\pi\colon X \to \Delta$ is of Type III and D_{ij} is smooth). We will show that this implies that $h^0(V_i, \mathcal{O}_{V_i}(D_{ij})) \geq 2$, so that D_{ij} moves in a linear system of dimension ≥ 1 and hence cannot be fixed.

First consider the case where V_i is elliptic ruled. Then Proposition 3.3.12 implies that $h^0(V_i, \mathcal{O}_{V_i}(D_{ij})) = D_{ij}^2 + 1 \geq 2$ when $D_{ij}^2 > 0$, and the lemma is proved in this case.

Next, consider the case when V_i is rational and $\pi\colon X \to \Delta$ is of Type II. Then an application of Proposition 3.3.7(a) with $C = D_{ij}$ irreducible and elliptic shows that $h^0(V_i, \mathcal{O}_{V_i}(D_{ij})) = D_{ij}^2 + 1 \geq 2$ when $D_{ij}^2 > 0$, proving the lemma in this case.

Third, consider the case when V_i is rational, $\pi\colon X \to \Delta$ is of Type III and D_{ij} is a rational nodal curve. Then, by [Fri83, Lemma 5], $h^0(V_i, \mathcal{O}_{V_i}(D_{ij})) \geq 1 + D_{ij}^2 \geq 2$ when $D_{ij}^2 > 0$, as required in this case.

Finally, consider the case when V_i is rational, $\pi\colon X \to \Delta$ is of Type III and D_{ij} is smooth. Then $D_{ij}.D_i = D_{ij}^2 + 2 > 0$ for $D_{ij}^2 \geq -1$, so we may apply Proposition 3.3.7(a) with $C = D_{ij}$ smooth, irreducible and rational to get $h^0(V_i, \mathcal{O}_{V_i}(D_{ij})) = D_{ij}^2 + 2 \geq 2$ exactly when $D_{ij}^2 \geq 0$. This completes the proof of the lemma. □

Now the fact that $(D_j|_{V_i})^2 = 0$ follows from Lemma 3.4.3, Lemma 3.4.4 and the triple point formula. So it just remains to show that $H_i.D_j = 0$. This will follow from:

Proposition 3.4.5. *Let V_k be the normalised components of the central fibre X_0 in a degeneration of K3 surfaces $\pi\colon X \to \Delta$ of Type II or III and let D_k denote the locus of double curves on V_k. Let $\mathcal{L}$ be a nef line bundle on X_0 such that $\mathcal{L}.\mathcal{L} = 2$ and there exist nonvanishing sections in $H^0(V_k, \mathcal{L})$ for all k. Let H_k denote a divisor defined on V_k by such a section. Then if an irreducible component $D_{ij} = V_i \cap V_j$ of D_i is in the fixed locus of $|H_i|$ for some i, it must satisfy $H_i.D_{ij} = 0$.*

Proof. Before we begin with the proof, we make a remark about non-normal components. If V_i is a non-normal component in X_0 it intersects itself along a smooth rational curve D_{ii}. When we normalise it we find that D_{ii} has two preimages. As these preimages will usually be considered alongside double curves that lie in two different components, we will abuse notation and refer to them as $D_{ij}|_{V_i}$ and $D_{ij}|_{V_j}$, where it is understood that if i and j are equal these refer to the disjoint curves in the normalisation. Finally, we note that in this case if one of these curves is in the fixed locus of $|H_i|$ then the other must also be.

We now proceed with the proof. Suppose D_{ij} is in the fixed locus of $|H_i|$. Then, using Lemma 3.4.3 if $i \neq j$, we see that D_{ij} is also in the fixed locus of $|H_j|$. So, by Lemma 3.4.4, $(D_{ij}|_{V_i})^2 \leq 0$ and $(D_{ij}|_{V_j})^2 \leq 0$, and these inequalities are strict when the degeneration $\pi\colon X \to \Delta$ is Type III and D_{ij} is smooth. Putting this into the triple point formula

$$\begin{aligned}(D_{ij}|_{V_i})^2 + (D_{ij}|_{V_j})^2 &= -T_{ijk} \\ &= \begin{cases} 0 & \text{if Type II or Type III with } D_{ij} \text{ nodal} \\ -2 & \text{if Type III with } D_{ij} \text{ smooth} \end{cases}\end{aligned}$$

we get that $(D_{ij}|_{V_i})^2 = (D_{ij}|_{V_j})^2 = 0$ in the first case and $(D_{ij}|_{V_i})^2 = (D_{ij}|_{V_j})^2 = -1$ in the second case.

Now write the linear system $|H_i|$ as

$$|H_i| = |H_i|_m + |H_i|_f,$$

where $|H_i|_f$ is the fixed part of $|H_i|$ and $|H_i|_m$ has no fixed components. Note that $|H_i|_m$ and $|H_i|_f$ are effective or trivial and $|H_i|_m$ is nef.

Suppose that we are on a component V_i with $H_i^2 = 0$. Then

$$0 = H_i^2 = |H_i|_m^2 + |H_i|_m.|H_i|_f + H_i.|H_i|_f.$$

As H_i and $|H_i|_m$ are nef, all terms on the right hand side of this equation are zero. Furthermore, by effectiveness of $|H_i|_f$, we have that $H_i.F = 0$ for all irreducible components F of $|H_i|_f$. Hence if D_{ij} is fixed in $|H_i|$, then $H_i.D_{ij} = 0$.

So we are left with the case where D_{ij} is the intersection of two components V_i and V_j with $H_i^2 > 0$ and $H_j^2 > 0$. As $\mathcal{L}.\mathcal{L} = 2$ on X_0, this can only occur if $H_i^2 = H_j^2 = 1$, or if $i = j$ and $H_i^2 = 2$ (in which case V_i is the normalisation of a surface that intersects itself). In these cases we can explicitly analyse the form of $|H_i|$ on V_i.

Let $|H_i|$ be a linear system on V_i, with $H_i^2 = 1$ or 2 and D_{ij} be a double curve that is a fixed component of $|H_i|$. As above, write

$$H_i^2 = |H_i|_m^2 + |H_i|_m.|H_i|_f + H_i.|H_i|_f. \tag{3.1}$$

If $H_i.|H_i|_f = 0$, we are done as above. So assume $H_i.|H_i|_f > 0$. We will show that this implies that $|H_i|_f^2 > 0$ and $|H_i|_m.|H_i|_f = 0$, then use these expressions to derive a contradiction.

If $H_i^2 = 1$, then $H_i.|H_i|_f = 1$ and so

$$1 = H_i.|H_i|_f = |H_i|_m.|H_i|_f + |H_i|_f^2.$$

As $|H_i|_m$ is nef, by equation (3.1) necessarily $|H_i|_m.|H_i|_f = 0$. So $|H_i|_f^2 = 1$.

If $H_i^2 = 2$ then necessarily $i = j$, so $D_{ij}|_{V_i}$ and $D_{ij}|_{V_j}$ both lie in V_i and so are both in $|H_i|_f$. In this case, write

$$|H_i|_f = a_0(D_{ij}|_{V_i}) + a_1(D_{ij}|_{V_j}) + \sum_{k=2}^{m} a_k F_k,$$

for irreducible curves F_k and integers a_k with $a_k > 0$. Then

$$H_i.|H_i|_f = a_0 H_i.(D_{ij}|_{V_i}) + a_1 H_i.(D_{ij}|_{V_j}) + \sum_{k=2}^{m} a_k H_i.F_k.$$

As H_i is nef, all of the terms in the right hand side of this equation are non-negative. Furthermore, we may assume $H_i.(D_{ij}|_{V_i}) > 0$, otherwise we are done. But we know $H_i.(D_{ij}|_{V_i}) = H_i.(D_{ij}|_{V_j})$, so $H_i.|H_i|_f \geq 2$. Therefore, using equation (3.1) and the fact that $|H_i|_m$ is nef, $H_i.|H_i|_f = 2$ and $|H_i|_m.|H_i|_f = 0$. Then as

$$2 = H_i.|H_i|_f = |H_i|_m.|H_i|_f + |H_i|_f^2,$$

we must have $|H_i|_f^2 = 2$.

In either case $|H_i|_f^2 > 0$ and $|H_i|_m.|H_i|_f = 0$. Note further that, for any component F of $|H_i|_f$, we must have $|H_i|_m.F = 0$ as $|H_i|_f$ is effective and $|H_i|_m$ is nef. So, for all components F of $|H_i|_f$,

$$|H_i|_f.F = |H_i|_m.F + |H_i|_f.F = H_i.F \geq 0$$

as H_i is nef. Therefore as $|H_i|_f$ is effective, it must be nef.

Now consider the case where V_i is elliptic ruled. Then $|H_i|_f^2 = 1$, as no component may intersect itself in a degeneration of Type II. By Lemma 3.3.9, $[|H_i|_f] = as + bf$ in $H^2(V_i, \mathbb{Z})$, where s is the class of a section and f is the class of a fibre. Furthermore, as $(D_{ij}|_{V_i})^2 = 0$, we have that $s^2 = 0$. So $1 = |H_i|_f^2 = 2ab$ by Lemma 3.3.9 again. But a and b are integers, so this cannot occur. Thus we obtain a contradiction, so $H_i.|H_i|_f = 0$ and we are done as above.

Finally, consider the case where V_i is rational. Then $|H_i|_f$ is effective, $|H_i|_f^2 > 0$ and $|H_i|_f.D_i \geq 0$ as $|H_i|_f$ is nef. So, by [Fri83, Lemma 5], $h^0(V_i, \mathcal{O}_{V_i}(|H_i|_f)) \geq 2$, contradicting $|H_i|_f$ being fixed. □

Applying this proposition with $\mathcal{L}$ equal to the restriction of $\mathcal{O}_X(H)$ to X_0 gives $H_i.D_j = 0$, as required. This completes the proof of Theorem 3.4.1. □

So let $\pi\colon X \to \Delta$ be a degeneration of Type II and let H be an effective, nef and π-flat divisor on X which induces a polarisation of degree two on the general fibre. Suppose that we have performed a series of elementary modifications on X so that the conclusions of Theorem 3.4.1 hold; by Lemma 2.3.7 this does not affect the form of the relative log canonical model. Furthermore, as H is still nef, the map $\phi\colon X \to X^c$ is still a morphism and by Lemma 3.2.5 the restriction of ϕ to X_0 agrees with the map $\phi_0\colon X_0 \to (X_0)^c$. To prove the Type II case of Theorem 3.2.2, all that remains is to calculate the image of $\phi_0\colon X_0 \to (X_0)^c$.

Let H_i denote the intersection of H with V_i. Note that H_i is effective and nef. The morphism

$$\phi_{V_i} \colon V_i \longrightarrow \operatorname{Proj} \bigoplus_{n \geq 0} H^0(V_i, \mathcal{O}_{V_i}(nH_i))$$

defined by the linear systems $|nH_i|$ for $n \geq 0$ agrees with the morphism defined by the restriction of ϕ_0 to V_i. We can gather information about these morphisms by studying the linear systems $|nH_i|$, and use this to deduce information about ϕ_0.

First, however, we will reduce the number of cases that we have to study. In order to do this, we use the theory of 0-, 1- and 2-surfaces developed in Section 3.3. We begin with a result classifying the configurations of 2-surfaces in X_0:

Lemma 3.4.6. *All of the 2-surfaces occurring in X_0 appear in a single configuration $\Sigma = \{V_0, \ldots, V_k\}$ where V_0 is rational, $V_1, \ldots, V_k$ are elliptic ruled and V_i meets $V_{i\pm 1}$ for $1 \leq i \leq k-1$. Furthermore, if $V_{k+1} \notin \Sigma$ is a surface intersecting V_k along the double curve $D_{k+1} = V_{k+1} \cap V_k$, then V_{k+1} is a 0-surface and $(D_{k+1}|_{V_{k+1}})^2 < 0$.*

Proof. [SB83b, Proposition 2.5] says that any 2-surfaces must occur in a configuration Σ as above and proves the result about V_{k+1}. Thus, it only remains to show that there is only one such configuration.

So suppose $X_0 = V_0 \cup \cdots \cap V_r$ and that $\{V_0, \ldots, V_k\}$ is a chain of 2-surfaces. If X_0 contains a second distinct chain of 2-surfaces, it must be $\{V_r, V_{r-1}, \ldots, V_m\}$, as the end surface in such a chain must be rational. Then, by [SB83b, Proposition 2.5], $(D_{k+1}|_{V_{k+1}})^2 < 0$ and $(D_m|_{V_{m-1}})^2 < 0$. But $V_{k+1}, \ldots, V_{m-1}$ are elliptic ruled so, by Lemma 3.3.10 and the triple point formula,

$$(D_{k+1}|_{V_{k+1}})^2 = -(D_{k+2}|_{V_{k+1}})^2 = (D_{k+2}|_{V_{k+2}})^2 = \cdots = -(D_m|_{V_{m-1}})^2$$

contradicting $(D_m|_{V_{m-1}})^2 < 0$. So there can be only one chain of 2-surfaces, and the proof is complete. ∎

Before continuing, it will prove beneficial to introduce a schematic representation that will be used in the remainder of this section. We can represent the configuration of surfaces in X_0 as a diagram

$$a_0 \,|\, a_1 \,|\, a_2 \,|\, \cdots \,|\, a_r \,,$$

where $a_i \in \{0, 1, 2\}$ represents an a_i-surface, and $|$ represents an elliptic double curve. Then, for instance, Lemma 3.4.6 says that any 2-surfaces in X_0 must occur in a configuration

$$2 \,|\, 2 \,|\, \cdots \,|\, 2 \,|\, 0 \,|\, \cdots \,.$$

With this in place, we prove another result that enables us to classify the 0-surfaces:

Lemma 3.4.7. *Any elliptic ruled* 0*-surface must intersect a* 2*-surface.*

Proof. Let V_i be an elliptic ruled 0-surface and let H_i denote the polarisation divisor on H_i. Let D_i and D_{i+1} denote the elliptic double curves on V_i. By Lemma 3.3.10, without loss of generality we may assume that $(D_i|_{V_i})^2 = e \geq 0$. So, in the notation of Lemma 3.3.9, in $H^2(V_i, \mathbb{Z})$ we have $[D_i] = s$, $[D_{i+1}] = s - ef$ and, by Lemma 3.3.11, $[H_i] = as + bf$ for $a > 0$ and $b \geq 0$.

Now consider $H_i^2 = a^2e + 2ab$. This is strictly positive by Lemma 3.3.2 and, as $(H|_{X_0})^2 = 2$, must equal 1 or 2.

If $H_i^2 = 1$, then necessarily $a = 1$, $b = 0$ and $e = 1$. In this case, $(D_{i+1}|_{V_i})^2 = -1$ and $H_i.D_{i+1} = 0$. Since $D_{i+1} = V_i \cap V_{i+1}$, by the triple point formula, $(D_{i+1}|_{V_{i+1}})^2 = 1$. But then D_{i+1} is an effective divisor on V_{i+1} with strictly positive self-intersection and $H_i.D_{i+1} = 0$ so, by Proposition 3.3.1, V_{i+1} is a 2-surface.

If $H_i^2 = 2$, one of two cases may occur. Either $a = 1$, $b = 0$ and $e = 2$ or $a = 1$, $b = 1$ and $e = 0$. In the first case, an argument identical to the one above gives that V_i must intersect a 2-surface. So we are left with the case $a = 1$, $b = 1$ and $e = 0$.

In this case, by Lemma 3.3.2, V_i is the only 0-surface. As $H_i.D_i = H_i.D_{i+1} = 1$, H_i cannot be numerically trivial on either of the components intersecting V_i, so both of these components must be 1-surfaces. Therefore, by Lemma 3.4.6, X_0 cannot contain any 2-surfaces, so must have the form

$$1 \,|\, \cdots \,|\, 1 \,|\, 0 \,|\, 1 \,|\, \cdots \,|\, 1\,,$$

where the 0 occurs in the ith place.

Now consider V_0. By the argument above, V_0 is a rational 1-surface with $H_0^2 = 0$. Furthermore, repeated application of Lemma 3.3.11, gives that $H_0.D_1 = H_i.D_i = 1$, where D_1 denotes the unique elliptic double curve on V_0. Finally, (V_0, D_1) is an anti-canonical pair by Lemma 3.3.6, so $K_{V_0} = -D_1$ and $H_0.(H_0 + K_{V_0}) = H_0^2 - H_0.D_1 = -1$. But the genus formula for effective divisors implies that this number must be even. This is a contradiction, so the second case cannot occur and V_i must intersect a 2-surface. □

We are now in a position to begin computing the morphisms ϕ_{V_i} and using them to calculate the image of ϕ_0. The results of Lemma 3.4.6 and Lemma 3.4.7 place strong restrictions on the possible configurations of 0-, 1- and 2-surfaces in X_0. It is easily seen that the only cases are:

(A) $0 \,|\, 1 \,|\, 1 \,|\, \cdots \,|\, 1 \,|\, 0$

(B) $0 \,|\, 1 \,|\, 1 \,|\, \cdots \,|\, 1 \,|\, 0 \,|\, 2 \,|\, \cdots \,|\, 2$

(C) $0 \,|\, 1 \,|\, 1 \,|\, \cdots \,|\, 1 \,|\, 1$

(D) $1 \,|\, 1 \,|\, \cdots \,|\, 1 \,|\, 0 \,|\, 2 \,|\, \cdots \,|\, 2$

(E) $0 \,|\, 2 \,|\, 2 \,|\, \cdots \,|\, 2 \,|\, 2$

Note that in cases (A) and (B) we allow there to be no 1-surfaces present.

We will consider each of the above cases in turn and apply the results of Section 3.3

to examine the linear systems $|nH_i|$ for $n \geq 0$. This will then be used to calculate the morphism $\phi_0 \colon X_0 \to (X_0)^c$.

We begin with case (A). In this case, V_0 and V_r are 0-surfaces with $H_0^2 = H_r^2 = 1$. We claim that $H_0.D_1 > 0$. To see why this is the case, suppose for a contradiction that $H_0.D_1 \leq 0$. Then, as H_0 is nef, $H_0.D_1 = 0$. Furthermore, by the Hodge Index Theorem, $(D_1|_{V_0})^2 < 0$ so, by the triple point formula, $(D_1|_{V_1})^2 > 0$. But then, by Proposition 3.3.1, V_1 would be a 2-surface, which is a contradiction.

Next, by Lemma 3.3.11, we know that H_i is a sum of fibres on each of the 1-surfaces $V_1, \ldots, V_{r-1}$, so ϕ_0 contracts each of these surfaces onto the double curve $D_1 \cong D_r$. Thus, we may restrict to the case where there are only the two surfaces V_0 and V_r meeting along an elliptic double curve D.

Given this setup, Friedman shows in the proof of [Fri84, Theorem 5.4] that there are two distinct possibilities:

(A1) H_i is connected for $i = 0, r$ with $p_a(H_0) = p_a(H_r) = 0$, $H_0^2 = H_r^2 = 1$ and $H_i.D = 3$.

(A2) H_i is connected for $i = 0, r$ with $p_a(H_0) = p_a(H_r) = 1$, $H_0^2 = H_r^2 = 1$ and $H_i.D = 1$.

By Lemma 3.3.6, we know that (V_0, D) and (V_r, D) are anticanonical pairs. Furthermore, Theorem 3.4.1 shows that the linear systems $|H_0|$ and $|H_r|$ do not contain D in their fixed loci. So we may use Theorem 3.3.5 to examine these linear systems.

Using this the first thing we note is that, by Theorem 3.3.5(4), the graded algebra $\bigoplus_{n \geq 0} H^0(V_i, \mathcal{O}_{V_i}(nH_i))$ is generated in degrees ≤ 3 for $i = 0, r$. This will enable us to use Proposition 3.3.7 to calculate the images of ϕ_{V_i}. We now proceed with the analysis of the first two cases:

(3.2.2) Case II.3. Suppose we are in case (A1). If $|H_i|$ has fixed components they must satisfy case (1.2) of Theorem 3.3.5, as $H_i.D > 0$. In this case the fixed part $|H_i|_f$ of $|H_i|$

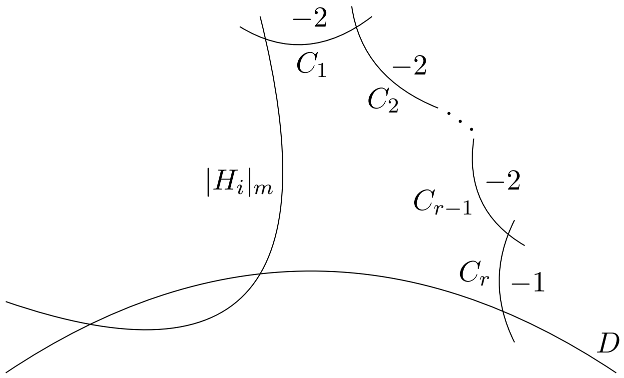

Figure 3.3.

has $|H_i|_f.D = 1$ by Lemma 3.3.4, so the mobile part $|H_i|_m$ has $|H_i|_m.D = 2$. Hence, the mobile part contains smooth irreducible members and we have the configuration shown in Figure 3.3.

Since $H_i.C_j = 0$ for all fixed components C_j, we contract these to give a linear system $|H_i'|$ with no fixed components. As $H_i'.D = 3$, we see that $|H_i'|$ also has no base points by Theorem 3.3.5(2). Hence, $|H_i|$ may be taken to be base point free, and contains smooth irreducible members. By Proposition 3.3.7(a), we have the following dimensions for the cohomology groups:

$$\begin{aligned} h^0(V_i, \mathcal{O}_{V_i}(H_i)) &= 3 \\ h^0(V_i, \mathcal{O}_{V_i}(2H_i)) &= 6 \\ h^0(V_i, \mathcal{O}_{V_i}(3H_i)) &= 10 \\ &\vdots \end{aligned}$$

Let x, y and z be generating sections of $H^0(V_i, \mathcal{O}_{V_i}(H_i))$. Since $|H_i|$ is base point free, the sections x^2, y^2, z^2, xy, xz and yz generate $H^0(V_i, \mathcal{O}_{V_i}(2H_i))$ and the ten degree 3 monomials generate $H^0(V_i, \mathcal{O}_{V_i}(3H_i))$. As noted above, by Theorem 3.3.5(4), the graded algebra $\bigoplus_{n\geq 0} H^0(V_i, \mathcal{O}_{V_i}(nH_i))$ is generated in degrees ≤ 3, so there will be no new monomials of higher weight. Continuing upwards, we find that there are no relations in higher degrees either.

Hence, ϕ_{V_i} is a morphism $\phi_{V_i}\colon V_i \to \mathbb{P}^2$ that takes D to a nonsingular cubic $\{f_3(x_i) = 0\} \subset \mathbb{P}^2[x_1, x_2, x_3]$. So ϕ_0 is a morphism from X_0 to a double cover of $\mathbb{P}^2$ ramified twice over the image of D:

$$\phi_0\colon X_0 \longrightarrow \{z^2 = f_3^2(x_i)\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z]$$

and we are in case (II.3a) of Theorem 3.2.2.

(3.2.2) Case II.4a. Now suppose we are in case (A2). If $|H_i|$ has fixed components, they must satisfy (1.2) of Theorem 3.3.5. In that case $|H_i|_m = kC$ for some irreducible curve C. Then $H_i^2 = |H_i|_m^2 + 2k - 1 = 1$, so we must have $k = 1$ and $|H_i|_m^2 = C^2 = 0$. Hence, by the genus formula

$$2p_a(C) - 2 = C^2 - C.D$$

and the fact that $|H_i|_m.D = H_i.D - 1 = 0$, we get that $p_a(C) = 1$, i.e. C is elliptic.

As in case (A1), we contract all fixed components to give a linear system $|H_i|$ without any. Furthermore, by Theorem 3.3.5(4), the linear system $|2H_i|$ is base point free. However, $H_i.D = 1$, so by Theorem 3.3.5(2) the linear system $|H_i|$ must have a single base point, where H_i meets D. Using Proposition 3.3.7(a), we find that ϕ_{V_i} is a morphism

$$\phi_{V_i}\colon V_i \longrightarrow \{z^2 - g_6^{(i)}(x_1, x_2, y) = 0\} \subset \mathbb{P}_{(1,1,2,3)}[x_1, x_2, y, z]$$

taking D to a curve $\{l^{(i)}(x_1, x_2) = 0\} \cap \phi_{V_i}(V_i)$. Since $H_i^2 > 0$, by the Hodge index theorem any curve E contracted by this morphism must have $E^2 < 0$. By Lemma 3.3.4, any such E must have $E^2 = -1$ or -2. Contracting such curves produces rational double points at worst, so $g_6^{(i)}(x_1, x_2, y) = 0$ has at worst A-D-E singularities. Note further that, as the image of D must be nonsingular, $g_6^{(i)}(0,0,1) \neq 0$.

To see how these glue to give the image of ϕ_0, write

$$\phi_{V_0}(V_0) = \{z^2 - g_6^{(0)}(x_1, x_2, y) = 0\} \subset \mathbb{P}_{(1,1,2,3)}[x_1, x_2, y, z],$$
$$\phi_{V_r}(V_r) = \{z^2 - g_6^{(r)}(x_1, x_3, y) = 0\} \subset \mathbb{P}_{(1,1,2,3)}[x_1, x_3, y, z].$$

Furthermore, after a linear change of coordinates in the x_i, we may assume that $\phi_{V_0}(D) = \{x_2 = 0\} \cap \phi_{V_0}(V_0)$ and $\phi_{V_r}(D) = \{x_3 = 0\} \cap \phi_{V_r}(V_r)$. So, for $i = 0, r$,

$$D = \{z^2 - g_6^{(i)}(x_1, 0, y) = 0\} \subset \mathbb{P}_{(1,2,3)}[x_1, y, z]$$

and $g_6^{(0)}(x_1, 0, y) = g_6^{(r)}(x_1, 0, y)$ (up to multiplication by a constant).

So, using this, write

$$g_6^{(0)}(x_1, x_2, y) = g_6^{(0)}(x_1, 0, y) + x_2 h_5^{(0)}(x_1, x_2, y),$$
$$g_6^{(r)}(x_1, x_3, y) = g_6^{(0)}(x_1, 0, y) + x_3 h_5^{(r)}(x_1, x_3, y).$$

Next set

$$f_6(x_1, x_2, x_3, y) = g_6^{(0)}(x_1, 0, y) + x_2 h_5^{(0)}(x_1, x_2, y) + x_3 h_5^{(r)}(x_1, x_3, y).$$

Then

$$\phi_{V_0}(V_0) = \{z^2 - f_6(x_1, x_2, x_3, y) = x_3 = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$
$$\phi_{V_r}(V_r) = \{z^2 - f_6(x_1, x_2, x_3, y) = x_2 = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$

so

$$\phi_0 \colon X_0 \longrightarrow \{z^2 - f_6(x_1, x_2, x_3, y) = x_2x_3 = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z].$$

Finally, note that as $g_6^{(i)}(0,0,1) \neq 0$ we must have $f_6(0,0,0,1) \neq 0$, so we are in case (II.4a) of Theorem 3.2.2.

(3.2.2) Case II.4b. Next we consider case (B). Let V_0 and V_k be the 0-surfaces, with $H_0^2 = H_k^2 = 1$. In the same way as case (A), we have that $H_0.D_1 > 0$ and ϕ_0 contracts the 1-surfaces $V_1, \ldots, V_{k-1}$ onto the double curve $D_1 \cong D_k$. Furthermore, as H is numerically trivial on a 2-surface, $V_{k+1}, \ldots, V_r$ are contracted to a point.

Now consider the linear system $|H_k|$ on V_k. As V_{k+1} is a 2-surface, $H_k.D_{k+1} = 0$. By the Hodge index theorem, $(D_{k+1}|V_k)^2 \leq 0$ and, by Lemma 3.3.10, $(D_k|_{V_k})^2 \geq 0$. Thus, using the notation of Lemma 3.3.9, in $H^2(V_k, \mathbb{Z})$ we may write $[D_k] = s$ and $[D_{k+1}] = s - (D_k|_{V_k})^2 f$. Furthermore, by Lemma 3.3.11, $[H_k] = as + bf$ for $a > 0$ and $b \geq 0$. Thus, using $H_k.D_{k+1} = b = 0$ and $H_k^2 = 1$, we get that $[H_k] = s$ and $s^2 = 1$.

Now, using the structure of the Picard group from Lemma 3.3.9, we have that $H_k \sim D_k$ on V_k. Furthermore, by Corollary 3.3.13, we know that the graded algebra $\bigoplus_{n \geq 0} H^0(V_k, \mathcal{O}_{V_k}(nH_k))$ is generated in degrees ≤ 3. Using this and Proposition 3.3.12, we see that ϕ_{V_k} is a morphism

$$\phi_{V_k} \colon V_k \longrightarrow \{z^2 - g_6^{(k)}(x_1, x_2, y) = 0\} \subset \mathbb{P}_{(1,1,2,3)}[x_1, x_2, y, z]$$

taking D_k to a curve $\{l^{(k)}(x_1, x_2) = 0\} \cap \phi_{V_k}(V_k)$ and contracting D_{k+1} to a point. Note that, as V_k is minimal, by [Har77, Proposition V.2.20], D_{k+1} is the only irreducible curve in V_k with negative self-intersection so, by the Hodge Index Theorem, is the only irreducible curve C with $H_i.C = 0$. Thus, D_{k+1} is the only curve in V_k contracted by ϕ_{V_k}. As D_{k+1} is elliptic with self-intersection (-1), this leads to a simple elliptic singularity of type $\tilde{E}_8$ in $\phi_{V_k}(V_k)$. By Proposition 3.1.8, such a singularity arises from a consecutive triple point in the branch curve $\{g_6^{(k)}(x_1, x_2, y) = 0\} \subset \mathbb{P}(1,1,2)$.

We next turn our attention to V_0. By the same argument as was used to prove case (A2), we see that ϕ_{V_0} is a morphism

$$\phi_{V_0} \colon V_0 \longrightarrow \{z^2 - g_6^{(0)}(x_1, x_2, y) = 0\} \subset \mathbb{P}_{(1,1,2,3)}[x_1, x_2, y, z],$$

where $\{g_6^{(0)}(x_1, x_2, y) = 0\} \subset \mathbb{P}(1,1,2)$ has at worst A-D-E singularities and D_1 is mapped to a curve $\{l^{(0)}(x_1, x_2) = 0\} \cap \phi_{V_0}(V_0)$.

Putting this together in the same way as case (A2), after a linear change of coordinates in the x_i we get

$$\phi_0 \colon X_0 \longrightarrow \{z^2 - f_6(x_1, x_2, x_3, y) = x_2 x_3 = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$

where $\{f_6(x_1, x_2, x_3, y) = x_j = 0\} \subset \mathbb{P}(1,1,1,2)$ has a consecutive triple point for exactly one choice of $j \in \{2, 3\}$ and at worst A-D-E singularities for the other.

Finally, note that as the image of D_1 under ϕ_{V_0} must be nonsingular, $g_6^{(0)}(0,0,1) \neq 0$. Thus, $f_6(0,0,0,1) \neq 0$ also, and we are in case (II.4b) of Theorem 3.2.2.

Now assume that we are in case (C). In this case, V_0 is a 0-surface with $H_0^2 = 2$. In the same way as case (A), we have that $H_0.D_1 > 0$ and ϕ_0 contracts the 1-surfaces $V_1, \ldots, V_{r-1}$ onto the double curve $D_1 \cong D_r$. Thus, we may restrict to the case where we have a 0-surface V_0 and a 1-surface V_r meeting along an elliptic double curve D.

Given this setup, Friedman shows in the proof of [Fri84, Theorem 5.4] that there are two distinct possibilities:

(C1) H_i is connected for $i = 0, r$ with $p_a(H_0) = 1$, $p_a(H_r) = 0$, $H_0^2 = 2$, $H_r^2 = 0$ and $H_i.D = 2$.

(C2) H_0 is connected but $|H_r|$ fails to have connected members. In this case $p_a(H_0) = 0$,

$H_0^2 = 2$, $H_r^2 = 0$ and $H_i.D = 4$.

As in case (A), we may use Theorem 3.3.5 to examine these linear systems. The first thing we note is that, by Theorem 3.3.5(4), the graded algebra $\bigoplus_{n\geq 0} H^0(V_i, \mathcal{O}_{V_i}(nH_i))$ is generated in degrees ≤ 3 for $i = 0, r$. This will enable us to use Proposition 3.3.7 to calculate the images of ϕ_{V_i}. We now proceed with the analysis of the individual cases:

(3.2.2) Case II.1a. First assume we that are in case (C1). In the same way as case (A1) we may arrange that $|H_0|$ has no fixed components, and Theorem 3.3.5(2) implies that it is base point free. Then Theorem 3.3.5(1.2) implies that $|H_0|$ contains irreducible members so, using Proposition 3.3.7(a), we find that ϕ_{V_0} is a morphism

$$\phi_{V_0}\colon V_0 \longrightarrow \{z^2 - f_4(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2)}[x_1, x_2, x_3, z]$$

where $f_4(x_i) = 0$ has at worst A-D-E singularities and D is mapped to a nonsingular curve $\{l(x_i) = 0\} \cap \phi_{V_0}(V_0)$ (which is a double cover of the rational curve $\{l(x_i) = z = 0\}$ ramified over four points).

Now we look at V_r. By Theorem 3.3.5(3.1), $|H_r|$ has no base locus and so ϕ_{V_r} is a morphism $\phi_{V_r}\colon V_r \to \mathbb{P}^1$ exhibiting V_r as a ruled surface. The restriction of ϕ_{V_r} to D exhibits D as a double cover of $\mathbb{P}^1$ ramified over four points.

Let ψ denote the morphism

$$\begin{aligned} \psi\colon \{z^2 - f_4(x_i) = 0\} &\longrightarrow \{w^2 - l^2(x_i) f_4(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, w] \\ (x_1 : x_2 : x_3 : z) &\longmapsto (x_1 : x_2 : x_3 : l(x_i) z) \end{aligned}$$

Then ψ is an isomorphism outside of D, and D is mapped $2{:}1$ onto the rational curve $\{l(x_i) = w = 0\}$. So ψ realises the contraction of V_r as restricted to V_0, and hence $\psi \circ \phi_{V_0}(V_0) = \phi_0(V_0)$. Therefore we are in case (II.1a) of Theorem 3.2.2.

(3.2.2) Case II.2. Next assume we are in case (C2). In the same way as before we may arrange that $|H_0|$ has no fixed components, and Theorem 3.3.5 implies that it is base point free and has irreducible members. Then, using Proposition 3.3.7(a), we find that ϕ_{V_0} is a morphism

$$\phi_{V_0} \colon V_0 \longrightarrow \{z^2 - f_2(x_i) = 0\} \subset \mathbb{P}^3[x_1, x_2, x_3, z]$$

where $f_2(x_i) = 0$ has at worst A-D-E singularities and D is mapped to a nonsingular curve $\{q(x_i) = 0\} \cap \phi_{V_0}(V_0)$ (which is a double cover of the rational curve $\{q(x_i) = z = 0\}$ ramified over four points).

Turning our attention to V_r, in the proof of [Fri84, Theorem 5.4] Friedman shows that V_r is a ruled surface with $H_r = 2F$, for F a fibre of the ruling. Since F is irreducible, we may use Proposition 3.3.7(a) to show that ϕ_{V_r} is a contraction

$$\phi_{V_r} \colon V_r \longrightarrow Q := \{q(x_i) = 0\} \subset \mathbb{P}^2[x_i]$$

with q a nonsingular quadric. The restriction of ϕ_{V_r} to D exhibits D as a double cover of Q ramified over four points.

Now, as before, we let ψ denote the morphism

$$\begin{aligned} \psi \colon \{z^2 - f_2(x_i) = 0\} &\longrightarrow \{w^2 - q^2(x_i) f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, w] \\ (x_1 : x_2 : x_3 : z) &\longmapsto (x_1 : x_2 : x_3 : q(x_i) z) \end{aligned}$$

Then ψ is an isomorphism outside of D, and D is mapped $2{:}1$ onto the rational curve $\{q(x_i) = w = 0\}$. So ψ realises the contraction of V_r as restricted to V_0, and hence $\psi \circ \phi_{V_0}(V_0) = \phi_0(V_0)$. Therefore we are in case (II.2) of Theorem 3.2.2.

(3.2.2) Case II.1b. Now assume that we are in case (D). Let $V_0, \ldots, V_{k-1}$ be the 1-surfaces, V_k be the 0-surface and $V_{k+1}, \ldots, V_r$ be the 2-surfaces. In the same way as before, $H_k.D_k > 0$ and ϕ_0 contracts the 1-surfaces onto the double curve $D_1 \cong D_k$.

Moreover, as H is numerically trivial on a 2-surface, $V_{k+1}, \dots, V_r$ are contracted to a point.

The same argument as was used in case (B) shows that $H_k.D_{k+1} = 0$, $(D_{k+1}|_{V_k})^2 \leq 0$ and $(D_k|_{V_k})^2 \geq 0$. So, in the notation of Lemma 3.3.9, in $H^2(V_k, \mathbb{Z})$ we can write $[D_k] = s$, $[D_{k+1}] = s - (D_k|_{V_k})^2 f$ and $[H_k] = as + bf$ for some $a > 0$ and $b \geq 0$. Then, using $H_k.D_{k+1} = b = 0$ and $H_k^2 = 2$, we get that $[H_k] = s$ and $s^2 = 2$.

Now, using the structure of the Picard group from Lemma 3.3.9, we have that $H_k \sim D_k$ on V_k. Furthermore, by Corollary 3.3.13, we know that the graded algebra $\bigoplus_{n \geq 0} H^0(V_k, \mathcal{O}_{V_k}(nH_k))$ is generated in degrees ≤ 3. Using this and Proposition 3.3.12, we see that ϕ_{V_k} is a morphism

$$\phi_{V_k} \colon V_k \longrightarrow \{z^2 - f_4(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2)}[x_1, x_2, x_3, z]$$

mapping D_k to a nonsingular curve $\{l(x_i) = 0\} \cap \phi_{V_0}(V_0)$ (which is a double cover of the rational curve $\{l(x_i) = z = 0\}$ ramified over four points) and contracting D_{k+1} to a point. In the same way as case (B), D_{k+1} is the only curve contracted by ϕ_{V_k}. As D_{k+1} is elliptic with self-intersection (-2), this leads to a simple elliptic singularity of type $\tilde{E}_7$ in $\phi_{V_k}(V_k)$. By Proposition 3.1.8, such a singularity arises from a quadruple point in the branch curve $\{f_4(x_i) = 0\} \subset \mathbb{P}^2$.

Now we turn our attention to V_0. As in case (C1), ϕ_{V_0} is a morphism $\phi_{V_0} \colon V_0 \to \mathbb{P}^1$ exhibiting V_0 as a ruled surface. The restriction of ϕ_{V_0} to D_1 exhibits D_1 as a double cover of $\mathbb{P}^1$ ramified over four points.

Let ψ denote the morphism

$$\begin{aligned} \psi \colon \{z^2 - f_4(x_i) = 0\} &\longrightarrow \{w^2 - l^2(x_i) f_4(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, w] \\ (x_1 : x_2 : x_3 : z) &\longmapsto (x_1 : x_2 : x_3 : l(x_i) z) \end{aligned}$$

Then ψ is an isomorphism outside of D_k, and D_k is mapped 2:1 onto the rational curve

$\{l(x_i) = w = 0\}$. So ψ realises the contraction of V_0 as restricted to V_k, and hence $\psi \circ \phi_{V_k}(V_k) = \phi_0(V_k)$. Therefore we are in case (II.1b) of Theorem 3.2.2.

Finally, we consider case (E). In this case, V_0 is a 0-surface with $H_0^2 = 2$ and $V_1, \ldots, V_r$ are 2-surfaces. As H is numerically trivial on $V_1, \ldots, V_r$, these surfaces are contracted to a point by ϕ_0, so we need only consider the restriction of ϕ_0 to V_0.

So consider V_0. Note that, by Lemma 3.3.6, (V_0, D_1) is an anticanonical pair and, by Theorem 3.4.1, D_1 is not fixed in the linear system $|H_0|$. So we may use Theorem 3.3.5 to examine these linear systems. We first note that Theorem 3.3.5(4) implies that the graded algebra $\bigoplus_{n \geq 0} H^0(V_0, \mathcal{O}_{V_0}(nH_0))$ is generated in degrees ≤ 3. This will enable us to use Proposition 3.3.7 to calculate the image of ϕ_{V_0}.

We now proceed with the analysis of the linear system $|H_0|$. There are 2 cases to consider:

(E1) $|H_0|$ has no fixed components.

(E2) $|H_0|$ has fixed components.

(3.2.2) Case II.0h. We first consider case (E1), where $|H_0|$ has no fixed components. Then, by Theorem 3.3.5(2), $|H_0|$ has no base points either. So, by Bertini's Theorem, $|H_0|$ contains irreducible members. Let $C \in |H_0|$ be one such. Then C is nef, $C^2 = 2$ and $C \cap D_1 = \emptyset$, so we may apply Proposition 3.3.7(b) to see that ϕ_{V_0} is a morphism

$$\phi_{V_0} \colon V_0 \longrightarrow \{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z].$$

This morphism contracts exactly those irreducible curves E in V_0 satisfying $H_0.E = 0$. By the Hodge Index Theorem, such curves must have $E^2 < 0$. So, by Lemma 3.3.4, if $E \neq D_1$ is contracted then E is rational with $E^2 = -1$ or -2. So ϕ_{V_0} contracts D_1 and a collection of (-1)- and (-2)-curves. This gives rise to an elliptic singularity, possibly along with some rational double points.

It just remains to classify the elliptic singularity arising in $\phi_{V_0}(V_0)$. Suppose that $Z = D_1 + \sum_{i=1}^{k} E_i$ is a maximal connected configuration of 0-curves in V_0 (recall that a 0-*curve* is an irreducible curve C with $H_i.C = 0$). By the argument above, E_i is a (-1)- or (-2)-curve for $1 \leq i \leq k$. As Z is connected, at least one of the E_i satisfies $E_i.D_0 = 1$, so has $E_i^2 = -1$ by Lemma 3.3.4. Thus, we can contract this E_i without affecting the nonsingularity of V_0. Furthermore, Z remains connected after performing this procedure, so we may iterate it to obtain V_0 nonsingular with $Z = D_1$. Given this situation, by Definition 3.1.7, D_1 contracts to give a simple elliptic singularity in $\phi_{V_0}(V_0)$. We will find the type of this singularity by calculating the self-intersection $(D_1|_{V_0})^2$ (which is strictly negative by the Hodge Index Theorem).

Note that $\dim |H_0| = 2$, so $|H_0|$ contains enough divisors to sweep out V_0. Therefore, there must exist an element of the form $(D_1 + E) \in |H_0|$ for some effective E. We claim that E is nef. To see this, let C be an irreducible curve in V_0. Then $E.C = H_0.C - D_1.C$. For a contradiction, we suppose that this is strictly negative for some C. Then, since E is effective, we must have $C^2 < 0$.

Note that since $E.D_1 = -(D_1|_{V_0})^2 > 0$, we may assume $C \neq D_1$. Furthermore, if $C.D_1 = 0$, then $E.C = H_0.C \geq 0$ as H_0 is nef. So $C.D_1 > 0$. Therefore, by Lemma 3.3.4, $C^2 = -1$ and $C.D_1 = 1$. But then in order for $C.E < 0$ to hold, we must have $H_0.C = 0$. So C is a (-1)-curve with $C.D_1 = 1$ and $H_0.C = 0$, which cannot exist as all such curves were contracted above.

Hence E is nef with

$$E^2 = H_0^2 - 2H_0.D_1 + (D_1|_{V_0})^2 = 2 + (D_1|_{V_0})^2 \geq 0.$$

So $-2 \leq (D_1|_{V_0})^2 < 0$, and $\phi_{V_0}(V_0)$ contains a simple elliptic singularity of type $\tilde{E}_7$ or $\tilde{E}_8$. By Proposition 3.1.8, such singularities arise respectively from quadruple points or consecutive triple points in the branch curve $\{f_6(x_i) = 0\} \subset \mathbb{P}^2$. Therefore we are in case (II.0h) of Theorem 3.2.2.

(3.2.2) Case II.0u. Finally, we consider case (E2), where $|H_0|$ has a fixed component. By Theorem 3.3.5(1.1), the fixed part $|H_0|_f = C$ is a smooth rational (-2)-curve and the mobile part $|H_0|_m = 2E$ for some smooth elliptic E with $E^2 = E.D_1 = 0$ and $C.E = 1$.

Let C and E be as above. Then since C is a rational (-2)-curve, Lemma 3.3.4 implies that $C.D_1 = 0$ and so, as C is irreducible, $C \cap D_1 = \emptyset$. Furthermore, E is an irreducible elliptic curve with $E \neq D_1$ and $E.D_1 = 0$, so $E \cap D_1 = \emptyset$ also. So the divisor $(C + 2E) \sim H_0$ satisfies the conditions of Lemma 3.3.7(b) and we may apply this Lemma to see that

$$h^0(V_0, \mathcal{O}_{V_0}(nH_0)) = n^2 + 2.$$

To find the map ϕ_{V_0} we note that $|H_0|$ has a fixed component, but Theorem 3.3.5(4) shows that $|nH_0|$ has no fixed components or base locus for $n \geq 2$. So there must be a quadric relation in $H^0(V_0, \mathcal{O}_{V_0}(2H_0))$, and ϕ_{V_0} defines a morphism

$$\phi_{V_0} \colon V_0 \longrightarrow \{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z].$$

This morphism contracts exactly those curves Z in V_0 with $H_0.Z = 0$. The (-2)-curve C is contracted to the point $(0:0:0:1:z)$ lying over the vertex of the cone $\{f_2(x_i) = 0\} \subset \mathbb{P}(1,1,1,2)$, and D_1 is contracted to a simple elliptic singularity. By a similar calculation to the one performed in case (E1), we find that the resulting surface has a simple elliptic singularity of type $\tilde{E}_7$ or $\tilde{E}_8$, along with some rational double points. By Proposition 3.1.8, these singularities arise from either a quadruple point or a consecutive triple point, possibly along with some A-D-E singularities, in the branch curve $\{f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}(1,1,1,2)$. Finally, note that since the elliptic curve E is nonsingular, this branch curve cannot pass through the vertex $(0:0:0:1)$ of the cone. Therefore we are in case (II.0u) of Theorem 3.2.2.

This completes the analysis of the Type II fibres.

3.5 Type III Fibres

It remains to classify the Type III fibres. We would like to use similar techniques to those we used in the Type II case to study $(X, \mathcal{L})$. In order to do this, we would like to prove a Type III analogue of Theorem 3.4.1.

As before, we begin by setting up some notation. Let $\pi\colon X \to \Delta$ be a degeneration of K3 surfaces with $X_0 = \pi^{-1}(0)$ a fibre of Type III. Write $X_0 = V_0 \cup \cdots \cup V_r$ for rational surfaces V_i. Note that we assume here that the surfaces V_i have been normalised. Let D_{ij} denote the rational double curve $V_i \cap V_j$ and let $D_i = \bigcup_j D_{ij}$ denote the double locus on V_i. Recall that a double curve D_{ij} is called a $(*)$-*curve* if it is nonsingular and has self-intersection (-1) on both components in which it lies. As usual, $\mathcal{L}$ denotes a line bundle that induces a polarisation of degree two on a general fibre and H denotes an effective, nef and π-flat divisor in the linear system defined by $\mathcal{L}$.

Remark 3.5.1. Before we embark on the results of this section, we make a remark on non-normal components. Suppose that $\overline{V}_i$ is a non-normal component in the Type III fibre X_0. Let $\nu\colon V_i \to \overline{V}_i$ denote the normalisation. Then the non-normal locus in $\overline{V}_i$ is a smooth rational curve $\overline{D}_{ii}$. The preimage $D_{ii} := \nu^{-1}(\overline{D}_{ii})$ consists of two disjoint rational curves in V_i. As these curves will often be considered alongside double curves that lie in two different components, we will frequently abuse notation and refer to them as $D_{ij}|_{V_i}$ and $D_{ij}|_{V_j}$, where it is understood that if i and j are equal these refer to the disjoint curves in the normalisation. Finally, when we refer to the restriction of a line bundle or divisor to a component V_i, we understand this to mean the pull-back of that restriction under the normalisation map ν.

We begin with the analogue of Theorem 3.4.1 alluded to above:

Theorem 3.5.2. *Let $\pi\colon X \to \Delta$ be a Type III degeneration of K3 surfaces over Δ and let H be an effective, nef and π-flat divisor on X that induces a polarisation of degree two on the generic fibre. Let H_i be the effective (or zero) divisor on V_i defined by intersecting with H. If a component D_{ij} of the double locus D_i is a fixed component of $|H_i|$, then D_{ij} is a $(*)$-curve with $H_i.D_{ij} = 0$.*

Proof. Suppose that the double curve $D_{ij} = V_i \cap V_j$ is fixed in the linear system $|H_i|$. Applying Proposition 3.4.5 with $\mathcal{L}$ equal to the restriction of $\mathcal{O}_X(H)$ to X_0, and noting that this restriction defines the complete linear system $|H_i|$ on each component V_i of X_0 by Lemma 3.2.5, we get that $H_i.D_{ij} = 0$, proving half of the theorem.

Next, using Lemma 3.4.3 we see that D_{ij} must also be fixed in $|H_j|$ (or, if $i = j$, we note that both components of D_{ii} must be fixed in $|H_i|$). This places a strong restriction on D_{ij}: by Lemma 3.4.4 it must have negative self-intersection in both V_i and V_j (or non-positive if D_{ij} is nodal). In this setting, the triple point formula gives

$$(D_{ij}|_{V_i})^2 + (D_{ij}|_{V_j})^2 = -T_{ijk} = \begin{cases} 0 & \text{if } D_{ij} \text{ is a nodal curve on } V_i \text{ or } V_j \\ -2 & \text{otherwise} \end{cases}$$

If D_{ij} is nonsingular, this implies that it must be a $(*)$-curve, as required. So it only remains to show that D_{ij} cannot be a rational nodal curve.

So suppose for a contradiction that $D_{ij}|_{V_i}$ is a rational nodal curve that is fixed in $|H_i|$. Then V_i cannot be a 2-surface, as H_i is numerically trivial on such surfaces. Moreover V_i cannot be a 1-surface, as D_{ij} is a 0-curve and by Proposition 3.3.1 the 0-curves on a 1-surface form a ruling, so we would have $D_{ij}|_{V_i} \cong \mathbb{P}^1$. This contradicts our assumption that $D_{ij}|_{V_i}$ is nodal. So V_i must be a 0-surface.

In order to reach a contradiction we will show that any 0-surface that satisfies $H_i.D_{ij} = 0$ for all of its double curves D_{ij} can border only 2-surfaces. Then, noting that D_{ij} is the only double curve in V_i, we must have that V_j is a 2-surface. But then, as H_j is numerically trivial, D_{ij} cannot be fixed in $|H_j|$. This contradicts Lemma 3.4.3.

Thus, it only remains to prove:

Lemma 3.5.3. *With assumptions as in Theorem 3.5.2, suppose in addition that V_i is a 0-surface in X_0. Then if $H_i.D_{ij} = 0$ for all double curves D_{ij} in V_i,*

(a) *V_i is the only 0-surface, and*

(b) *all of the other components of X_0 are 2-surfaces.*

Proof. We begin with (a). Note that, $H_i^2 > 0$ by Lemma 3.3.2, so H_i^2 must equal 1 or 2. Then, by the Riemann-Roch Theorem,

$$\Big(H_i^2 + \sum_j H_i.D_{ij}\Big) \in 2\mathbb{Z}.$$

If $H_i^2 = 1$ then $H_i.D_{ij} > 0$ for some j, as H_i is nef, and we are done. So we may assume $H_i^2 = 2$. By Lemma 3.3.2 again, V_i can be the only 0-surface in X_0.

Now consider (b). We want to show that X_0 cannot contain any 1-surfaces. Suppose that this is not the case, so that there exists a 1-surface V_{j_1} in X_0. By Proposition 3.3.1 we see that V_{j_1} is ruled by the pencil of 0-curves on it. Let F_{j_1} be an irreducible curve in this pencil that is not a component of the double curve D_{j_1}. Then, by Lemma 3.3.4, $F_{j_1}.D_{j_1} = 2$. As D_{j_1} is effective, this implies that D_{j_1} contains either two sections or one bisection of the ruling for V_{j_1}. Let $D_{j_1j_2} = V_{j_1} \cap V_{j_2}$ be one such section or bisection.

Now consider the divisor H_{j_1} on V_{j_1} defined by intersecting with H. By Lemma 3.3.2 we have $H_{j_1}^2 = 0$ so, as H_{j_1} is effective and nef, H_{j_1} must be a sum of 0-curves. Thus H_{j_1} is a sum of fibres of the ruling for V_{j_1} and $H_{j_1}.D_{j_1j_2} > 0$ as $D_{j_1j_2}$ is a section or bisection of this ruling.

Next consider the component V_{j_2}. As $H_{j_2}.D_{j_1j_2} = H_{j_1}.D_{j_1j_2} > 0$, this component cannot be a 2-surface. Moreover it cannot be a 0-surface, as by assumption V_i is the only 0-surface and $H_i.D_{ik} = 0$ for all k. Therefore, V_{j_2} must be another 1-surface.

Repeating the argument above, we see that H_{j_2} must be a sum of fibres of a ruling for V_{j_2} and $D_{j_1j_2}$ must be a section or bisection of that ruling. Furthermore, if $D_{j_1j_2}$ is a section then there is another double curve $D_{j_2j_3}$ on V_{j_2} that is a section of the same ruling, so we may repeat the process to find a 1-surface V_{j_3} meeting V_{j_2} along $D_{j_2j_3}$. If $D_{j_1j_2}$ is a bisection then no other double curves on V_{j_2} are sections or bisections of the ruling, so the process terminates.

Repeating this argument as many times as possible and relabeling components if necessary, we obtain either:

- a cycle $V_{j_1}, \ldots, V_{j_n}$ of 1-surfaces meeting along sections of a given ruling for each; or
- a chain $V_{j_1}, \ldots, V_{j_n}$ of 1-surfaces such that $D_{j_kj_{k+1}} = V_{j_k} \cap V_{j_{k+1}}$ is a section of a given ruling on V_{j_k} for $k \in \{2, \ldots, n-1\}$ and a bisection of a given ruling on V_1 and V_n.

Moreover $H_{j_k}.F_{j_k} = 0$ for any fibre F_{j_k} of the given ruling on V_{j_k} and any $k \in \{1, \ldots, n\}$. However, such configurations of 1-surfaces are excluded by [SB83b, Lemma 2.2]. This is a contradiction, so X_0 cannot contain any 1-surfaces. □

This completes the proof of Theorem 3.5.2. □

So let H be defined as before. By Lemma 3.2.5 the restriction of the morphism $\phi\colon X \to X^c$ to X_0 agrees with the morphism $\phi_0\colon X_0 \to (X_0)^c$. To prove the Type III case of Theorem 3.2.2, all that remains is to calculate the image of $\phi\colon X_0 \to (X_0)^c$.

As in Theorem 3.5.2, let H_i denote the intersection of H with V_i. Note that H_i is effective and nef. The morphism

$$\phi_{V_i}\colon V_i \longrightarrow \operatorname{Proj} \bigoplus_{n \geq 0} H^0(V_i, \mathcal{O}_{V_i}(nH_i))$$

defined by the linear systems $|nH_i|$ for $n \geq 0$ agrees with the map defined by the restriction of ϕ_0 to V_i. We can gather information about these maps by studying the linear systems $|nH_i|$, and use this to deduce information about ϕ_0.

The purpose of Theorem 3.5.2 then becomes clear. It means that ϕ_0 contracts any double curves D_{ij} that are fixed in $|H_i|$ for some i, j. Furthermore, as these D_{ij} are $(*)$-curves, this contraction does not affect the nonsingularity of V_i or V_j (although the total space X may become singular). This allows us to get rid of such fixed double curves, so that we can use Theorem 3.3.5 to study the components of X_0.

Following a similar method to the Type II case, we now wish to show that the 1- and 2-surfaces in X_0 are contracted by ϕ_0. This will allow us to restrict to the case where we have (at most) two components meeting along a union of rational curves. We have:

Lemma 3.5.4. *If V_i is a* 1*-surface* (*resp.* 2*-surface*)*, then V_i is contracted to a curve* (*resp. point*) *by ϕ_{V_i}.*

Proof. First note that, by the argument above, we may assume that no double curve in V_i is fixed in $|H_i|$. Furthermore, as V_i is not a 0-surface, by Lemma 3.3.2 we know that $H_i^2 = 0$.

Thus, if V_i is a 1-surface, we may apply Theorem 3.3.5 to see that $H_i \sim kC$ for some $k > 0$ and C irreducible with $C^2 = 0$. Furthermore, as C is a 0-curve, by Proposition 3.3.1 we know that C is a fibre of a ruling for V_i. So H_i is a sum of fibres in V_i and ϕ_{V_i} contracts V_i to a section of the ruling.

Finally, if V_i is a 2-surface, then H_i is numerically trivial. So $H_i.E = 0$ for any effective divisor E on V_i, and ϕ_{V_i} contracts V_i to a point. □

In order to deal with the other components we note first that if V_i is a 0-surface, Lemma 3.3.2 shows that it must have $H_i^2 > 0$. So there can exist at most 2 such components. Furthermore, by Theorem 3.3.5(4), on a 0-surface the graded algebra

$\bigoplus_{n\geq 0} H^0(V_i, \mathcal{O}_{V_i}(nH_i))$ is generated in degrees ≤ 3. This will prove very useful for calculating the maps ϕ_{V_i}.

With this in place, we can proceed with the classification. There are two cases to consider; the case where there are two components with $H_i^2 > 0$ (in which case $H_i^2 = 1$ on each) and the case where there is exactly one component with $H_i^2 > 0$ (in which case $H_i^2 = 2$).

So assume that there are two components in X_0 with $H_i^2 = 1$. Without loss of generality, we may renumber the components so that these are V_0 and V_1. By Lemma 3.5.4, ϕ_0 contracts all components except V_0 and V_1, leaving V_0 and V_1 meeting along a union of rational curves.

By the genus formula

$$2p_a(H_i) = 3 - H_i.D_i.$$

Since $H_i^2 > 0$, the linear system $|H_i|$ contains connected members and $p_a(H_i) \geq 0$. So $H_i.D_i$ is equal to 1 or 3. This gives 3 separate cases:

(3.2.2) Case III.3. First consider the case where $H_0.D_0 = H_1.D_1 = 3$. By an analogous argument to that used to prove case (II.3) of Theorem 3.2.2, we see that ϕ_0 is a morphism from X_0 to a double cover of $\mathbb{P}^2$ ramified over a singular cubic:

$$\phi_0 \colon X_0 \longrightarrow \{z^2 = f_3^2(x_i)\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z]$$

where $f_3(x_i) = 0$ has nodal singularities.

This proves the statement of case (III.3) in Theorem 3.2.2. Note however that there are 3 distinct subcases, distinguished by whether H_i intersects one, two or three rational components of D_i. These are best explained by Figure 3.4, which shows the locus $\{f_3(x_i) = 0\} \subset \mathbb{P}^2$ over which the double cover $\phi_0(X_0)$ of $\mathbb{P}^2$ is ramified with index two. This locus is also isomorphic to the image of $\phi_0(D_0)$ under the projection defining the double cover.

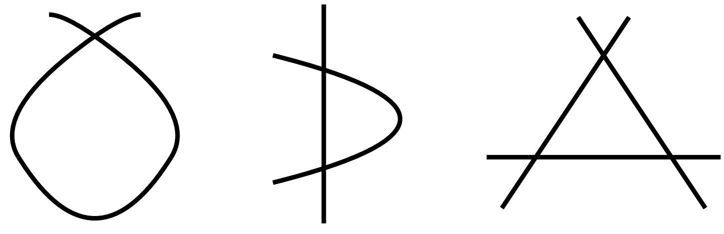

Figure 3.4.

In Figure 3.4(i) H_i meets exactly one component of D_i, so all other components are contracted by ϕ_0 and $f_3(x_i) = 0$ is a nodal cubic. Figure 3.4(ii) corresponds to when H_i meets exactly two components of D_i, here $f_3(x_i) = 0$ is the union of a line and a quadric. Finally, in Figure 3.4(iii) H_i meets three components of D_i and $f_3(x_i) = 0$ is a union of three lines.

(3.2.2) Case III.4a. Next, consider the case where $H_0.D_0 = H_1.D_1 = 1$. Again, by an analogous argument to that used to prove case (II.4a) of Theorem 3.2.2 we see that ϕ_{V_i} defines a morphism

$$\phi_{V_i}\colon V_i \longrightarrow \{z^2 - g_6^{(i)}(x_1, x_2, y) = 0\} \subset \mathbb{P}_{(1,1,2,3)}[x_1, x_2, y, z]$$

where $g_6^{(i)}(x_1, x_2, y) = 0$ has at worst A-D-E singularities. D_i is mapped to a curve $\{l^{(i)}(x_1, x_2) = 0\} \cap \phi_{V_i}(V_i)$ that meets $g_6^{(i)}(x_1, x_2, y) = 0$ at two distinct points, one of which has multiplicity two. Furthermore, by Theorem 3.3.5, the general member of the linear system $|H_i|$ is nonsingular, so the point $(0:0:1)$ cannot lie in the branch curve $\{g_6^{(i)}(x_1, x_2, y) = l^{(i)}(x_1, x_2) = 0\}$.

After a linear change of co-ordinates in the x_i, we see that ϕ_0 is a morphism from X_0 to a double cover of this image ramified twice over the image of D_i

$$\phi_0\colon X_0 \longrightarrow \{z^2 - f_6(x_1, x_2, x_3, y) = x_2 x_3 = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z].$$

Furthermore, as $g_6^{(i)}(0,0,1) \neq 0$, we have $f_6(0,0,0,1) \neq 0$, and we are in case (III.4a) of Theorem 3.2.2.

(3.2.2) Case III.4b. Lastly, we consider the case where $H_0.D_0 = 3$ and $H_1.D_1 = 1$. Note that this can only occur if H_0 intersects three components of D_0 and V_0 intersects itself along two of these components.

By the arguments above, ϕ_{V_0} defines a morphism $\phi_{V_0} \colon V_0 \to \mathbb{P}^2$, taking D_0 to a union of three lines. Furthermore, ϕ_{V_1} defines a morphism

$$\phi_{V_1} \colon V_1 \longrightarrow \{z^2 - g_6(x_1,x_2,y) = 0\} \subset \mathbb{P}_{(1,1,2,3)}[x_1,x_2,y,z]$$

where $g_6(x_1,x_2,y) = 0$ has at worst A-D-E singularities and $g_6(0,0,1) \neq 0$. After a linear change in coordinates in the x_i, we may assume that D_1 is mapped to a curve $\{x_2 = 0\} \cap \phi_{V_1}(V_1)$ that meets $g_6(x_1,x_2,y) = 0$ at two distinct points, one of which has multiplicity two.

Next, note that the restriction of ϕ_0 to V_0 is the composition of the map ϕ_{V_0} above with the morphism ψ identifying two of the components of $\phi_{V_0}(D_0)$ and taking the third to the curve

$$\{z^2 - g_6(x_1,0,y) = 0\} \subset \mathbb{P}_{(1,2,3)}[x_1,y,z].$$

By the description of g_6 above, after a quadratic change of coordinates in y we may factorise $g_6(x_1,0,y)$ as

$$g_6(x_1,0,y) = y^2(y - a^2x_1^2),$$

with $a \in \mathbb{C} - \{0\}$ (note that this change of coordinates in y is not strictly required in order to prove Theorem 3.2.2; we do it solely to make the calculations below easier to follow). Then if we assume, without loss of generality, that $\phi_{V_0}(D_0)$ is the union of the three hyperplane sections $\{x_i = 0\} \subset \mathbb{P}^2$, the morphism ψ maps $\mathbb{P}^2$ to the cone over the curve

$$\{z^2 - y^2(y - a^2x_1^2) = 0\} \subset \mathbb{P}_{(1,2,3)}[x_1, y, z].$$

Explicitly, the morphism ψ is given by

$$\begin{aligned}\psi\colon \mathbb{P}^2[x_1, x_2, x_3] &\longrightarrow \{z^2 - y^2(y - a^2x_1^2) = 0\} \subset \mathbb{P}_{(1,1,2,3)}[x_1, x_3, y, z]\\ (x_1 : x_2 : x_3) &\longmapsto \left(\frac{x_1 - x_2}{2a} : x_3 : x_1x_2 : x_1x_2(x_1 + x_2)\right).\end{aligned}$$

Note that ψ is an isomorphism outside of $\{x_1 = 0\}$ and $\{x_2 = 0\}$, which are identified. Moreover, it takes $\{x_3 = 0\}$ to the curve

$$\{z^2 - y^2(y - a^2x_1^2) = 0\} \subset \mathbb{P}_{(1,2,3)}[x_1, y, z],$$

as required.

Finally, the argument from case (II.4a) shows that we may find $f_6(x_1, x_2, x_3, y)$ such that

$$\begin{aligned}f_6(x_1, x_2, 0, y) &= g_6(x_1, x_2, y)\\ f_6(x_1, 0, x_3, y) &= y^2(y - a^2x_1^2).\end{aligned}$$

Thus, ϕ_0 is a morphism

$$\phi_0\colon X_0 \longrightarrow \{z^2 - f_6(x_1, x_2, x_3, y) = x_2x_3 = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z].$$

Furthermore, as $g_6(0, 0, 1) \neq 0$, we have $f_6(0, 0, 0, 1) \neq 0$, and we are in case (III.4b) of Theorem 3.2.2.

Our next task is to study the case where there is exactly one component with $H_i^2 > 0$. Without loss of generality, we may renumber this component V_0. Necessarily $H_0^2 = 2$.

By Lemma 3.5.4, we may contract all components except V_0. By the genus formula

$$2p_a(H_0) = 4 - H_0.D_0$$

where D_0 is the double curve on V_0. This gives that $H_0.D_0 = 0$, 2 or 4. We start with the cases where $H_0.D_0 > 0$, as the case where $H_0.D_0 = 0$ will require some extra work.

(3.2.2) Case III.1. Consider first the case where $H_0.D_0 = 2$. By an analogous argument to that used to prove case (II.1a) of Theorem 3.2.2, we see that ϕ_{V_0} is a morphism

$$\phi_{V_0} \colon V_0 \longrightarrow \{z^2 - f_4(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2)}[x_1, x_2, x_3, z]$$

where $f_4(x_i) = 0$ has at worst A-D-E singularities. D_0 maps to $\{l(x_i) = 0\} \cap \phi_{V_0}(V_0)$, where $l(x_i) = 0$ meets $f_4(x_i) = 0$ in two or three points, with multiplicities ≤ 2. The image of D_0 under this map is a double cover of the nonsingular curve $\{l(x_i) = z = 0\}$, ramified four times over these two or three points. As before, the contraction of the other components of X_0 maps the image of D_0 $2:1$ onto this curve. This proves case (III.1) of Theorem 3.2.2.

As before there are two subcases, distinguished by whether H_0 intersects one or two rational components of D_0. Suppose first that H_0 intersects exactly one rational component of D_0. Then ϕ_{V_0} contracts all but this component, so maps D_0 to a nodal cubic. This must be a double cover of $l(x_i) = 0$ ramified four times over three points (the only node occurs over the point with ramification index two). So $\phi_{V_0}(D_0)$ meets $f_4(x_i) = 0$ in three points, with multiplicities 1, 1 and 2. Then $\phi_0(X_0)$ is a double cover of $\mathbb{P}^2$ ramified over the configuration shown in Figure 3.5(i), where the index of ramification is one along the thinner curve and two along the thicker one.

Now suppose that H_0 intersects two rational components of D_0. Then ϕ_{V_0} maps D_0 to a pair of rational curves meeting in two nodes. This must occur as a double cover

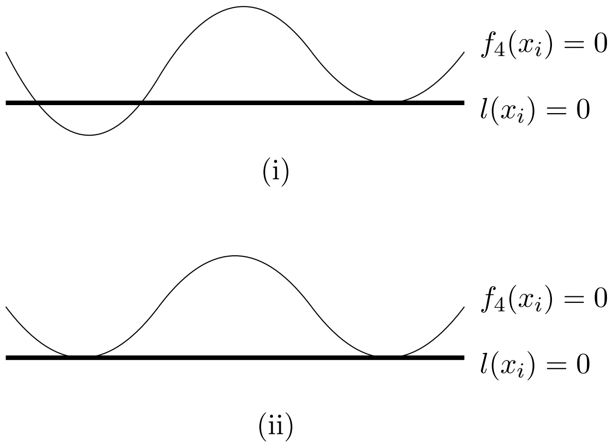

Figure 3.5.

of $l(x_i) = 0$ ramified four times over two points, with each point having ramification index two. Hence $\phi_{V_0}(D_0)$ meets $f_4(x_i) = 0$ in two points, each having multiplicity two. $\phi_0(X_0)$ is a double cover of $\mathbb{P}^2$ ramified over the configuration shown in Figure 3.5(ii) where, as above, the index of ramification is one along the thinner curve and two along the thicker one.

(3.2.2) Case III.2. Next consider the case where $H_0.D_0 = 4$. Again, by an analogous argument to that used to prove case (II.2) of Theorem 3.2.2, we see that ϕ_{V_0} is a morphism:

$$\phi_{V_0} \colon V_0 \longrightarrow \{z^2 - f_2(x_i) = 0\} \subset \mathbb{P}^3[x_1, x_2, x_3, z]$$

where $f_2(x_i) = 0$ has at worst A-D-E singularities. D_0 maps to $\{q(x_i) = 0\} \cap \phi_{V_0}(V_0)$, where $q(x_i) = 0$ meets $f_2(x_i)$ in two, three or four points, with multiplicities ≤ 2. The image of D_0 under ϕ_{V_0} is a double cover of the rational curve $\{q(x_i) = z = 0\}$ (that may have nodal singularities) ramified four times over these points. As before, the contraction of the other components of X_0 maps the image of D_0 $2\colon 1$ onto this curve. This proves case (III.2) of Theorem 3.2.2.

Once again there are five subcases, distinguished by the number of components of

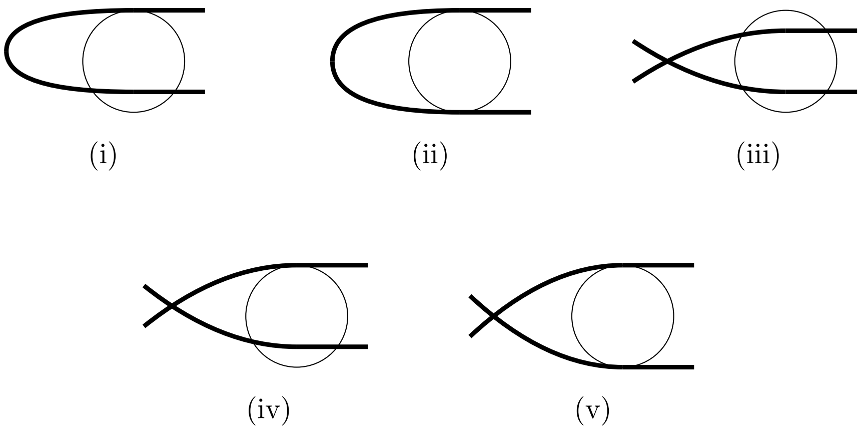

Figure 3.6.

D_0 that H_0 intersects. These cases are illustrated by Figure 3.6. In each case the thin circle represents the locus $f_2(x_i) = 0$ and the thick lines represent the locus $q(x_i) = 0$. As before, $\phi_0(X_0)$ is a double cover of $\mathbb{P}^2$ ramified over such a configuration, with ramification index two along the thick lines and one along the thin ones. The image of D_0 under ϕ_{V_0} is a double cover of the thick locus, ramified over its intersections with the circle. We will briefly analyse each case.

In Figure 3.6(i) H_0 intersects exactly one component of D_0, all other components are contracted. ϕ_{V_0} maps D_0 to a double cover of a quadric curve ramified over three points, one of them doubly. This image is a nodal cubic, with node at the point where the thick locus lies tangent to the circle.

In Figure 3.6(ii) H_0 intersects two components of D_0 and ϕ_{V_0} maps D_0 to a double cover of a nonsingular quadric ramified doubly at each of two points. In both this case and the next the image of D_0 is a pair of quadrics meeting at two points.

In Figure 3.6(iii) H_0 also intersects two components of D_0. In this case ϕ_{V_0} maps D_0 to a double cover of a pair of lines ramified over four points, two on each line. Note that this is the only subcase of (III.2) that has four distinct points of intersection between $\{q(x_i) = 0\}$ and $\{f_2(x_i) = 0\}$, and that in this case q has a nodal singularity.

In Figure 3.6(iv) H_0 intersects three components of D_0. The image of D_0 under ϕ_{V_0} is a double cover of a pair of lines ramified over three points, with two single ramification points on one line and a double one on the other. This image is a pair of lines (split from the line with the double ramification point) and a quadric (a double cover of a line ramified at two points), arranged to form a triangle.

Finally, in Figure 3.6(v) H_0 intersects four components of D_0. The image of D_0 under ϕ_{V_0} is a double cover of a pair of lines ramified doubly at two points, one on each line. Each of these lines thus splits into a pair of lines, making the image of D_0 into a configuration of four lines arranged to form a quadrilateral.

(3.2.2) Cases III.0h and III.0u. Finally, consider the case where $H_0.D_0 = 0$. Since H_0 is nef and $D_0 = \sum_i D_{0i}$ is effective, $H_0.D_{0i} = 0$ for all i. So D_0 is contracted by ϕ_{V_0}. Furthermore, Lemma 3.5.3 and [SB83b, Proposition 2.5] show that all of the other components of X_0 are 2-surfaces, so are contracted to a point along with D_0. Given this, in a manner analogous to case (II.0) we split into two subcases depending upon whether $|H_0|$ has fixed components or not.

In the case where $|H_0|$ has no fixed components, an argument similar to that used to prove case (II.0h) of Theorem 3.2.2 shows that ϕ_{V_0} is a morphism

$$\phi_{V_0} \colon V_0 \longrightarrow \{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z]$$

that contracts D_0 and a collection of (-1)- and (-2)-curves. This gives rise to some rational double points and a cusp singularity in the image surface.

In the case where $|H_0|$ has fixed components, an argument similar to that used to prove case (II.0u) of Theorem 3.2.2 shows that ϕ_{V_0} is a morphism

$$\phi_{V_0} \colon V_0 \longrightarrow \{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$

where $f_6(0,0,0,1) \neq 0$ and ϕ_{V_0} contracts D_0 to a cusp singularity and a collection of (-1)- and (-2)-curves to rational double points.

All that remains is to classify the cusp singularities occurring in the image of ϕ_0. By Proposition 3.1.9, in order to do this we just need to find the self-intersections of the components D_{0i} of the double curve D_0.

As in case (II.0h) of Theorem 3.2.2 we may contract any 0-curves that meet D_0 but are not contained in D_0, without affecting the nonsingularity of V_0 or the type of singularity obtained.

Furthermore, we now show that for all i we may assume

$$(D_{0i}|_{V_0})^2 \leq \begin{cases} -1 & \text{if } D_{0i}|_{V_0} \text{ is nodal} \\ -2 & \text{otherwise.} \end{cases}$$

By the Hodge Index Theorem, we know that $(D_{0i}|_{V_0})^2 < 0$. If $D_{0i}|_{V_0}$ is nodal then we are done. So assume that $D_{0i}|_{V_i}$ is smooth and rational. Then if $(D_{0i}|_{V_0})^2 = -1$ for some i, it must be a (-1)-curve. But then, as $H_0.D_{0i} = 0$, we may contract this (-1)-curve without affecting the nonsingularity of V_0 or the type of singularity obtained. Iterating this procedure, we may assume either that $D_{0i}|_{V_0}$ is nodal or $(D_{0i}|_{V_0})^2 \leq -2$ for all i.

Next we prove a lemma:

Lemma 3.5.5. *In the setting described above, there exists an element of $|H_0|$ of the form $E + D_0$, where E is effective and nef on V_0.*

Proof. Note first that $\dim |H_0| = 2$, so $|H_0|$ contains enough divisors to sweep out V_0. Choose some irreducible component D_{0i} of D_0. As $H_0.D_{0i} = 0$, there exists a connected element of $|H_0|$ of the form $D_{0i} + E$, for E effective or trivial. By the Hodge index theorem $(D_{0i}|_{V_0})^2 < 0$, so $E.D_{0i} > 0$. Let D_{0j} be another component of D_0 with

$D_{0j}.D_{0i} = 1$. Then

$$0 = H_0.D_{0j} = (D_{0i} + E).D_{0j} = 1 + E.D_{0j}.$$

So $E.D_{0j} < 0$ and, as E is effective, D_{0j} must be a component of E. Noting that D_0 is connected, repeat this process for all other components of D_0 to get an element of $|H_0|$ of the form $\sum_i D_{0i} + E$, for E effective or trivial.

It remains to show that E is nef. Let C be an irreducible curve on V_0. Then

$$E.C = H_0.C - D_0.C = H_0.C - \sum_i D_{0i}.C. \tag{3.2}$$

Suppose first that C is a double curve, $C = D_{0j}$ for some j. Then

$$E.C = E.D_{0j} = -\sum_i D_{0i}.D_{0j}.$$

If D_{0j} is nodal then D_{0j} is the only double curve and $(D_{0j}|_{V_0})^2 \leq 0$, so $E.D_{0j} \geq 0$ as required. Otherwise $-\sum_i D_{0i}.D_{0j} = -2 - (D_{0j}|_{V_0})^2$ and as $(D_{0j}|_{V_0})^2 \leq -2$, we have $E.D_{0j} \geq 0$ again. So we are done in this case.

Next suppose that C is not double, but that $C.D_{0i} = 0$ for all i. Then by equation (3.2), $E.C = H_0.C \geq 0$.

Thus we are left with the case where C is not double, but has $C.D_{0i} > 0$ for some i. Suppose that there exists such a curve with $E.C < 0$. Then, since E is an effective divisor on V_0, we must have $C^2 < 0$. So, by Lemma 3.3.4, C has $C^2 = -1$ and $\sum_i D_{0i}.C = 1$. By equation (3.2), we get that $H_0.C = 0$. But then C would be a 0-curve meeting D_0 but not contained in D_0, contradicting the fact that all such curves were contracted earlier.

Hence there are no irreducible curves C in V_0 with $E.C < 0$, so E is nef as required. $\square$

Let E be defined as in the lemma. Then

$$E^2 = H_0^2 - 2H_0.D_0 + (D_0|_{V_0})^2 = 2 + (D_0|_{V_0})^2 \geq 0,$$

which gives $(D_0|_{V_0})^2 \geq -2$. Combining this with the Hodge Index Theorem and the fact that

$$(D_{0i}|_{V_0})^2 \leq \begin{cases} -1 & \text{if } D_{0i}|_{V_0} \text{ is nodal} \\ -2 & \text{otherwise,} \end{cases}$$

we see that D_0 is either:

- A rational nodal curve with $(D_0|_{V_0})^2 = -1$ or -2; or
- A cycle of nonsingular rational curves $D_0 = \bigcup_i D_{0i}$, with $(D_0|_{V_0})^2 = -1$ or -2 and $(D_{0i}|_{V_0})^2 \leq -2$ for all i.

Thus, D_0 must be either a $(-1, r)$-cycle, a $(-2, r)$-cycle or a $(-1, -1, r_1, r_2)$-cycle, which give rise to the corresponding cusp singularities detailed in Proposition 3.1.9.

Remark 3.5.6. Note that almost all of the cusp singularities listed in Proposition 3.1.9 cannot possibly occur, as they require the branch curve to have degree greater than 6. There should be a direct way to prove this, by showing that the corresponding configurations of double curves cannot occur in a Type III degeneration of K3 surfaces of degree two.

In the case where D_0 is a $(-1, r)$-cycle and $r \geq 0$ is small, a long argument (that will not be reproduced here) using Looijenga's explicit classification of anticanonical rational surfaces [Loo81, Theorem 1.1] shows that if V_0 has such a cycle and admits a polarisation with $H_0^2 = 2$, then D_0 must be fixed in the linear system $|H_0|$. This contradicts Theorem 3.5.2 and shows that such a cycle cannot exist. Similar arguments show that in the case of a $(-2, r)$-cycle we get the same contradiction for $r \geq 2$ small, and for a $(-1, -1, r_1, r_2)$-cycle we get this contradiction for $r_1 \geq 1$ or $r_2 \geq 1$ and r_1, r_2

small. However, a direct proof that these singularities cannot occur for general (higher) values of r is still elusive.

This completes the analysis of the Type III fibres and the proof of Theorem 3.2.2.

Chapter 4

Constructing the Relative Log Canonical Model

4.1 Structure of the Relative Log Canonical Algebra

The aim of this chapter is to construct an explicit model for the relative log canonical model of a semistable terminal threefold fibred by K3 surfaces of degree two. In order to do this, we will try to emulate the construction of the canonical model for a fibration by genus 2 curves, given originally by Catanese and Pignatelli [CP06]. As such, the course of our construction will follow [CP06] quite closely.

We begin by recalling the set up. Let S denote a nonsingular complex curve and let $(X, \pi, \mathcal{L})$ be a semistable terminal threefold fibred by K3 surfaces of degree two over S. By the results of Section 2.4, after twisting the polarisation by $\mathcal{O}_X(Z)$, for some divisor Z supported on finitely many fibres of π, we may assume that the invertible sheaf $\mathcal{L}$ is locally π-flat.

Given such a pair $(X, \mathcal{L})$, by the results of Section 2.4 the relative log canonical model X^c of X is well-defined, and the classification of the fibres of X^c given in Theorem 3.2.2 holds. Unless otherwise specified, for the remainder of this chapter we will assume that

$(X, \mathcal{L})$ satisfies these properties.

By definition, the general fibre X_s of $\pi\colon X \to S$ is a K3 surface of degree two with polarisation induced by $\mathcal{L}_{X_s}$, the restriction of $\mathcal{L}$ to X_s. Furthermore, by Example 1.1.4, $\mathcal{L}_{X_s}$ is generated by its global sections and the general fibre X_s is hyperelliptic.

We are now ready to start our pursuit of an explicit construction for the relative log canonical model of $(X, \mathcal{L})$. Recall from Chapter 2 that the *relative log canonical algebra* of the pair $(X, \mathcal{L})$ is defined to be the graded algebra

$$\mathcal{R}(X, \mathcal{L}) = \bigoplus_{n=0}^{\infty} \mathcal{E}_n := \bigoplus_{n=0}^{\infty} \pi_*(\omega_X^n \otimes \mathcal{L}^n).$$

The relative log canonical algebra is useful because, by Theorem 2.4.9, the relative log canonical model of $(X, \mathcal{L})$ is equal to $\mathbf{Proj}_S(\mathcal{R}(X, \mathcal{L}))$. We will try to find a way to construct $\mathcal{R}(X, \mathcal{L})$ explicitly, which will in turn allow us to construct the relative log canonical model. First, however, we would like to know more about the structure of $\mathcal{R}(X, \mathcal{L})$.

We begin by noting that, by Lemma 1.3.4, $\mathcal{E}_n$ is a locally free $\mathcal{O}_S$-module for all $n \geq 0$.

Next, since the general fibre of X is isomorphic to a double cover of $\mathbb{P}^2$, there exists a birational involution ι on X "swapping the sheets", i.e. this involution extends the natural involution $(x_1, x_2, x_3, z) \mapsto (-x_1, -x_2, -x_3, z)$ on a general fibre $\{z^2 = f_6(x_i)\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x_2, x_3, z]$. We can use this involution to split the relative log canonical algebra into an invariant and an antiinvariant part. Let $U' \subset S$ be an open set. Then $U := \pi^{-1}(U')$ is ι-invariant and ι acts linearly on the space of sections $H^0(U, \omega_X^n \otimes \mathcal{L}^n) = \mathcal{E}_n(U)$, which splits as the direct sum of the $(+1)$-eigenspace and the (-1)-eigenspace.

This allows us to decompose $\mathcal{E}_n$ into

$$\mathcal{E}_n = \mathcal{E}_n^+ \oplus \mathcal{E}_n^-,$$

where $\mathcal{E}_n^+$ denotes the ι-invariant part and $\mathcal{E}_n^-$ denotes the ι-antiinvariant part. Using this, we can correspondingly split the relative log canonical algebra as

$$\mathcal{R}(X, \mathcal{L}) = \mathcal{R}(X, \mathcal{L})^+ \oplus \mathcal{R}(X, \mathcal{L})^-.$$

Furthermore, observe that $\mathcal{R}(X, \mathcal{L})^+$ is a subalgebra of $\mathcal{R}(X, \mathcal{L})$, and that $\mathcal{R}(X, \mathcal{L})^-$ is an $\mathcal{R}(X, \mathcal{L})^+$-module.

This decomposition will turn out to be invaluable when we attempt to construct $\mathcal{R}(X, \mathcal{L})$. We can calculate the ranks of the locally free sheaves $\mathcal{E}_n^+$ and $\mathcal{E}_n^-$ for $n \geq 1$ to get the following table:

n	rank $\mathcal{E}_n^+$	rank $\mathcal{E}_n^-$
even	$\frac{(n+1)(n+2)}{2}$	$\frac{(n-1)(n-2)}{2}$
odd	$\frac{(n-1)(n-2)}{2}$	$\frac{(n+1)(n+2)}{2}$

Furthermore, we know that $\mathcal{E}_0 = \mathcal{E}_0^+ = \mathcal{O}_S$ and $\mathcal{E}_1 = \mathcal{E}_1^-$.

Next we wish to study the multiplicative structure of $\mathcal{R}(X, \mathcal{L})$, paying particular attention to how it interacts with the decomposition above. Let $\mu_{n,m} \colon \mathcal{E}_n \otimes \mathcal{E}_m \to \mathcal{E}_{n+m}$ and $\sigma_n \colon \mathrm{Sym}^n(\mathcal{E}_1) \to \mathcal{E}_n$ denote the homomorphisms induced by multiplication in $\mathcal{R}(X, \mathcal{L})$. The maps σ_n will prove to be particularly useful as, if we can determine more information about them, we should be able to use them to reconstruct the sheaves $\mathcal{E}_n$ from $\mathcal{E}_1$. We have:

Lemma 4.1.1. *The maps $\sigma_n \colon \mathrm{Sym}^n(\mathcal{E}_1) \to \mathcal{E}_n$ are injective for all $n \geq 1$ and their image is contained in $\mathcal{E}_n^+$ when n is even and in $\mathcal{E}_n^-$ when n is odd.*

Proof. We begin by showing injectivity. Note that it is enough to show that the induced map of sections

$$\sigma_n(U)\colon H^0(U, \mathrm{Sym}^n(\mathcal{E}_1)) \longrightarrow H^0(U, \mathcal{E}_n)$$

is injective for all open sets $U \subset S$. So let $U \subset S$ be any open set. Then let $U' \subset U$ be the dense open subset over which the fibres of π are K3 surfaces of degree two. We first show that the induced map of sections $\sigma_n(U')$ on U' is injective, then use this to deduce the injectivity of $\sigma_n(U)$.

To show that $\sigma_n(U')$ is injective, it is enough to show that the induced maps on the fibres of the associated vector bundles

$$(\sigma_n)_s\colon \mathrm{Sym}^n(\mathcal{E}_1)_s \otimes_{\mathcal{O}_s} k(s) \longrightarrow (\mathcal{E}_n)_s \otimes_{\mathcal{O}_s} k(s)$$

are injective for all closed points $s \in U'$. Choose any such closed point s.

Let X_s denote the fibre of $\pi\colon X \to S$ over s and $\mathcal{L}_s$ be the invertible sheaf induced on X_s by $\mathcal{L}$. Since $\omega_{X_s} \cong \mathcal{O}_{X_s}$ and $\mathcal{L}_s$ is ample, by Serre duality and Kodaira vanishing $h^1(X_s, \mathcal{L}_s^n) = 0$ for all $s \in U'$. So, by the theorem on cohomology and base change [Mum70, Corollary II.5.2], we have isomorphisms

$$(\mathcal{E}_n)_s \otimes_{\mathcal{O}_s} k(s) \cong H^0(X_s, \mathcal{L}_s^n).$$

Now, since X_s is a K3 surface of degree two with polarisation divisor $\mathcal{L}_s$, Example 1.1.4 shows that we have a natural injection

$$\mathrm{Sym}^n H^0(X_s, \mathcal{L}_s) \hookrightarrow H^0(X_s, \mathcal{L}_s^n)$$

which induces $(\sigma_n)_s$ under the above isomorphism. So $(\sigma_n)_s$ is injective for all $s \in U'$, and hence $\sigma_n(U')$ is injective.

Finally, since $\mathrm{Sym}^n(\mathcal{E}_1)$ and $\mathcal{E}_n$ are both torsion free sheaves, the restriction maps

$H^0(U, \mathrm{Sym}^n(\mathcal{E}_1)) \to H^0(U', \mathrm{Sym}^n(\mathcal{E}_1))$ and $H^0(U, \mathcal{E}_n) \to H^0(U', \mathcal{E}_n)$ are both injective. So the map $\sigma_n(U)\colon H^0(U, \mathrm{Sym}^n(\mathcal{E}_1)) \to H^0(U, \mathcal{E}_n)$ must be injective, and hence so is σ_n.

With this in place, the statement on the images of σ_n follows immediately from the fact that $\mathcal{E}_1 = \mathcal{E}_1^-$. □

Define $\mathcal{T}_n := \mathrm{coker}(\sigma_n)$ and, using Lemma 4.1.1, write

$$\mathcal{T}_n^+ := \mathrm{coker}(\mathrm{Sym}^n(\mathcal{E}_1) \to \mathcal{E}_n^+) \quad \text{for } n \text{ even}$$
$$\mathcal{T}_n^- := \mathrm{coker}(\mathrm{Sym}^n(\mathcal{E}_1) \to \mathcal{E}_n^-) \quad \text{for } n \text{ odd.}$$

Then, by Lemma 4.1.1 again, we can decompose

$$\mathcal{T}_n = \begin{cases} \mathcal{T}_n^+ \oplus \mathcal{E}_n^- & \text{for } n \text{ even} \\ \mathcal{T}_n^- \oplus \mathcal{E}_n^+ & \text{for } n \text{ odd.} \end{cases}$$

Finally, note that the sheaves $\mathcal{T}_n^\pm$ are torsion sheaves.

With this in place, we are ready to begin describing how to construct $\mathcal{R}(X, \mathcal{L})$.

4.2 Constructing the Relative Log Canonical Algebra

In this section we embark on the explicit construction of the relative log canonical algebra $\mathcal{R}(X, \mathcal{L})$. In order to do this we follow the construction given by Catanese and Pignatelli in [CP06]. This will involve constructing a graded subalgebra $\mathcal{A}$ of $\mathcal{R}(X, \mathcal{L})$ that is simpler to construct explicitly, and that can act as a "stepping stone" on the way to the construction of $\mathcal{R}(X, \mathcal{L})$.

Before we start, however, it is convenient to explain some of the geometry that motivates this algebraic approach. As we have mentioned before, the general fibre of $\pi\colon X \to S$ is a double cover of $\mathbb{P}^2$ ramified over a smooth sextic curve. The construction of the K3-Weierstrass model studied in Chapter 1 can be thought of as constructing a $\mathbb{P}^2$-bundle over a Zariski open set $S_0 \subset S$, taking a double cover of this ramified over a divisor that intersects the general fibre in a smooth sextic, then completing across the gaps. However, as we saw in Example 2.1.1, if $\pi\colon X \to S$ contains any unigonal fibres this completion process causes bad singularities to appear in the K3-Weierstrass model.

To explain how we will to solve this problem, we need to be a little more precise about what is going wrong. Over the dense open set S_0, the K3-Weierstrass model can be seen as a double cover of the $\mathbb{P}^2$-bundle on S_0 given by $\mathbf{Proj}_{S_0}(\mathrm{Sym}(\mathcal{E}_1))$. The branch divisor is defined using the cokernel of the map $\sigma_3\colon \mathrm{Sym}^3(\mathcal{E}_1) \to \mathcal{E}_3$, which is locally free on S_0. Unfortunately, we find that if we try to extend this definition to all of S then we lose the local freeness of the cokernel, so the branch divisor is no longer well-defined. For this reason, we are forced to perform the construction on S_0 and complete to S.

This problem only occurs on fibres where the cokernel of the map σ_3 is not locally free. As we saw above, this cokernel can be written as $(\mathcal{T}_3^- \oplus \mathcal{E}_3^+)$, where $\mathcal{T}_3^-$ is a torsion sheaf. Furthermore, as we shall see in Lemma 4.2.1 below, $\mathcal{T}_3^-$ is supported exactly on the points of S corresponding to the unigonal fibres. This explains why the K3-Weierstrass construction fails on these fibres.

To solve this problem, we will construct an algebra $\mathcal{A}$ that takes better account of the properties of the maps σ_n than $\mathrm{Sym}(\mathcal{E}_1)$ does. Instead of a $\mathbb{P}^2$-bundle, $\mathbf{Proj}_S(\mathcal{A})$ will be a fibration of S by rational surfaces. We can then try to construct $X^c = \mathbf{Proj}_S(\mathcal{R}(X,\mathcal{L}))$ as a double cover of $\mathbf{Proj}_S(\mathcal{A})$.

However, in order to do this we will need to better understand the maps σ_n. We begin by studying the structure of the cokernels $\mathcal{T}_n$. We have the following analogue of [CP06, Lemma 4.1]:

Lemma 4.2.1. *Let $(X,\pi,\mathcal{L})$ be a semistable terminal threefold fibred by K3 surfaces of degree two. Suppose further that $\mathcal{L}$ is locally π-flat. Then*

(1) *$\mathcal{T}_2 = \mathcal{T}_2^+$ is isomorphic to the structure sheaf of an effective divisor τ, supported*

on the points of S corresponding to the unigonal fibres of π;

(2) *τ determines all the sheaves $\mathcal{T}_n$ as follows:*

$$\mathcal{T}_{2n}^{+} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{i\tau}^{\oplus(4(n-i)+1)}$$
$$\mathcal{T}_{2n+1}^{-} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{i\tau}^{\oplus(4(n-i)+3)}$$

Proof. (Following the proof of [CP06, Lemma 4.1]) By Theorem 3.2.2, there are two possibilities for the log canonical ring of a fibre of $\pi \colon X \to S$:

- The fibre is hyperelliptic and its log canonical ring is isomorphic to

$$\mathbb{C}[x_1, x_2, x_3, z]/(z^2 - f_6(x_i)),$$

where $\deg(x_i) = 1$ and $\deg(z) = 3$.

- The fibre is unigonal and its log canonical ring is isomorphic to

$$\mathbb{C}[x_1, x_2, x_3, y, z]/(z^2 - g_6(x_i, y),\ g_2(x_i)),$$

where $\deg(x_i) = 1$, $\deg(y) = 2$ and $\deg(z) = 3$. Furthermore, as $g_6(0,0,0,1) \neq 0$, we may assume that the coefficient of y^3 in g_6 is non-zero and, by completing the square in the x_i, we may also assume that

$$g_2(x_i) = x_1^2 - x_2(ax_2 + bx_3),$$

for some $a, b \in \mathbb{C}$ not both zero.

With this characterisation, we note that x_i are the ι-antiinvariant sections and y and z are ι-invariant. Furthermore, examination of these rings shows that the cokernels $\mathcal{T}_n$ are locally free away from the points whose fibres are unigonal, so the torsion sheaves

$\mathcal{T}_{2n}^{+}$ and $\mathcal{T}_{2n+1}^{-}$ are supported on these points.

Now, around a point P whose fibre is unigonal, $\mathcal{E}_2^{+}$ is locally generated by the sections x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3 and y. Then, by flatness, if t is a uniformising parameter for $\mathcal{O}_{S,P}$, we can lift the relation g_2 to

$$g_2(t) = x_1^2 - x_2(ax_2 + bx_3) + t\mu(t)y + t\psi(x_i, t).$$

Note that $\mu(t)$ is not identically zero, as x_1, x_2 and x_3 are algebraically independent for $t \neq 0$. Therefore, after changing coordinates in S, we may assume that $\mu(t) = t^{r-1}$ for a suitable integer $r \geq 1$. We call r the *multiplicity* of the point P. Using this and the relation above, the stalk of $\mathcal{T}_2$ at P is the $\mathcal{O}_{S,P}$-module

$$\mathcal{T}_{2,P} = \big(\mathrm{coker}(\sigma_2)\big)_P \cong \mathcal{E}_{2,P}^{+}/\mathrm{im}(\sigma_{2,P}) \cong \mathcal{O}_{S,P}/(t^r),$$

generated by the class of y.

Define τ to be the divisor on S given by $\sum_i r_i P_i$, where P_i are the points in S over which the fibres are unigonal and the r_i are the corresponding multiplicities. Then the stalk $\mathcal{O}_{\tau,P_i} \cong \mathcal{O}_{S,P_i}/(t^{r_i})$ and $\mathcal{T}_2 \cong \mathcal{O}_\tau$. This proves part (1) of Lemma 4.2.1.

Next, we can also choose a lifting of g_6 of the form

$$g_6(t) = z^2 - g_6'(x_i, y, t).$$

Since g_6 is ι-invariant, $g_6(t)$ must be also, otherwise z would vanish identically on the fibre over P. By flatness, $g_2(t)$ and $g_6(t)$ are all the relations of the stalk of $\mathcal{R}(X, \mathcal{L})$ at P.

First consider $\mathcal{T}_{2n+1}^{-}$. Its stalk at P is given by

$$(\mathcal{T}_{2n+1}^{-})_P = \big(\mathrm{coker}(\sigma_{2n+1})\big)_P \cong \mathcal{E}_{2n+1,P}^{-}/\mathrm{im}(\sigma_{2n+1,P}).$$

$\mathcal{E}^-_{2n+1,P}$ is generated by the $2n^2+5n+3$ monomials

$$\{x_1 h_{2n}(x_2,x_3,y),\ h_{2n+1}(x_2,x_3,y)\},$$

where $h_i(x_2,x_3,y)$ denotes any monomial of degree i in x_2, x_3 and y. Similarly, $\mathrm{im}(\sigma_{2n+1,P})$ is generated by the $4n+3$ monomials

$$\{x_1 h_{2n}(x_2,x_3),\ h_{2n+1}(x_2,x_3)\}.$$

So $(\mathcal{T}^-_{2n+1})_P$ is generated by the $2n^2+n$ monomials

$$\{x_1 y h_{2n-2}(x_2,x_3,y),\ y h_{2n-1}(x_2,x_3,y)\}.$$

These monomials can be listed as

$$\begin{aligned}
\{y^n x_i\} &\text{ generates } \mathcal{O}^{\oplus 3}_{n\tau,P}\\
\{x_1 y^{n-1} h_2(x_2,x_3),\ y^{n-1} h_3(x_2,x_3)\} &\text{ generates } \mathcal{O}^{\oplus 7}_{(n-1)\tau,P}\\
\{x_1 y^{n-2} h_4(x_2,x_3),\ y^{n-2} h_5(x_2,x_3)\} &\text{ generates } \mathcal{O}^{\oplus 11}_{(n-2)\tau,P}\\
&\ \ \vdots\\
\{x_1 y h_{2n-2}(x_2,x_3),\ y h_{2n-1}(x_2,x_3)\} &\text{ generates } \mathcal{O}^{\oplus (4n-1)}_{\tau,P}
\end{aligned}$$

and we see that $\mathcal{T}^-_{2n+1} \cong \bigoplus_{i=1}^n \mathcal{O}^{\oplus(4(n-i)+3)}_{i\tau}$.

By a similar calculation, the stalk of $\mathcal{T}^+_{2n}$ at P is given by

$$(\mathcal{T}^+_{2n})_P = \big(\mathrm{coker}(\sigma_{2n})\big)_P \cong \mathcal{E}^+_{2n,P}/\mathrm{im}(\sigma_{2n,P}).$$

$\mathcal{E}^+_{2n,P}$ is generated by the $2n^2+3n+1$ monomials

$$\{x_1 h_{2n-1}(x_2,x_3,y),\ h_{2n}(x_2,x_3,y)\},$$

and $\mathrm{im}(\sigma_{2n+1,P})$ is generated by the $4n+1$ monomials

$$\{x_1 h_{2n-1}(x_2,x_3),\ h_{2n}(x_2,x_3)\}.$$

So $(\mathcal{T}^+_{2n})_P$ is generated by the $2n^2-n$ monomials

$$\{x_1 y h_{2n-3}(x_2,x_3,y),\ y h_{2n-2}(x_2,x_3,y)\}.$$

These monomials can be listed as

$$\begin{aligned}
\{y^n\} &\text{ generates } \mathcal{O}^{\oplus 1}_{n\tau,P}\\
\{x_1 y^{n-1} h_1(x_2,x_3),\ y^{n-1} h_2(x_2,x_3)\} &\text{ generates } \mathcal{O}^{\oplus 5}_{(n-1)\tau,P}\\
\{x_1 y^{n-2} h_3(x_2,x_3),\ y^{n-2} h_4(x_2,x_3)\} &\text{ generates } \mathcal{O}^{\oplus 9}_{(n-2)\tau,P}\\
&\ \vdots\\
\{x_1 y h_{2n-3}(x_2,x_3),\ y h_{2n-2}(x_2,x_3)\} &\text{ generates } \mathcal{O}^{\oplus(4n-3)}_{\tau,P}
\end{aligned}$$

and we see that $\mathcal{T}^+_{2n} \cong \bigoplus_{i=1}^n \mathcal{O}^{\oplus(4(n-i)+1)}_{i\tau}$.

This completes the proof of Lemma 4.2.1. □

Using Lemma 4.2.1, if we know $\mathcal{T}_2$ we can determine all of the cokernels $\mathcal{T}_n$. So it seems sensible to expect that the structure of $\mathcal{R}(X,\mathcal{L})$ might be determined by its structure in low degrees. With this in mind, we define:

Definition 4.2.2. *Let $\mathcal{A}$ be the graded subalgebra of $\mathcal{R}(X,\mathcal{L})$ generated by $\mathcal{E}_1$ and $\mathcal{E}_2$.*

Let $\mathcal{A}_n$ denote its graded part of degree n and write

$$\mathcal{A} = \mathcal{A}_{\text{even}} \oplus \mathcal{A}_{\text{odd}} = \Big(\bigoplus_{n=0}^{\infty} \mathcal{A}_{2n}\Big) \oplus \Big(\bigoplus_{n=0}^{\infty} \mathcal{A}_{2n+1}\Big).$$

We similarly decompose $\mathcal{R}(X, \mathcal{L}) = \mathcal{R}_{\text{even}} \oplus \mathcal{R}_{\text{odd}}$. Then we have the following analogue of [CP06, Lemma 4.3]:

Lemma 4.2.3. *$\mathcal{R}(X, \mathcal{L})$ is isomorphic to $\mathcal{A} \oplus (\mathcal{A}[-3] \otimes \mathcal{E}_3^+)$ as a graded $\mathcal{A}$-module. Furthermore, $\mathcal{A}_{\text{even}}$ is the ι-invariant part of $\mathcal{R}_{\text{even}}$ and $\mathcal{A}_{\text{odd}}$ is the ι-antiinvariant part of $\mathcal{R}_{\text{odd}}$.*

Proof. (Following the proof of [CP06, Lemma 4.3]) We can unify the hyperelliptic and unigonal cases from the proof of Lemma 4.2.1 by writing the log canonical ring of a hyperelliptic fibre as

$$\mathbb{C}[x_1, x_2, x_3, y, z]/(y,\ z^2 - f_6(x_i)),$$

where the x_i are ι-antiinvariant of degree 1, and y and z are ι-invariant with degrees 2 and 3 respectively. Then in both cases the stalk of $\mathcal{A}$ is the subalgebra generated by x_1, x_2, x_3 and y, so $\mathcal{A}_{\text{even}}$ is ι-invariant and $\mathcal{A}_{\text{odd}}$ is ι-antiinvariant.

In both cases, locally on S we may write

$$\mathcal{R}(X, \mathcal{L}) \cong \mathcal{O}_S[x_1, x_2, x_3, y, z]/(f_2(t), f_6(t))$$

with $f_6(t) = z^2 - f_6'(x_i, y, t)$, so locally we have

(1) $\mathcal{A} \cong \mathcal{O}_S[x_1, x_2, x_3, y]/(f_2(t))$, and

(2) $\mathcal{R}(X, \mathcal{L}) \cong \mathcal{A} \oplus z\mathcal{A}$.

As z is a local generator of $\mathcal{E}_3^+$, this gives $\mathcal{R}(X, \mathcal{L}) \cong \mathcal{A} \oplus (\mathcal{A}[-3] \otimes \mathcal{E}_3^+)$.

Finally, the statement on the ι-invariant and ι-antiinvariant parts follows from the fact that $\mathcal{R}_{\text{even}} \cong \mathcal{A}_{\text{even}} \oplus z\mathcal{A}_{\text{odd}}$ and $\mathcal{A}_{\text{even}}$ is ι-invariant, and $\mathcal{R}_{\text{odd}} \cong \mathcal{A}_{\text{odd}} \oplus z\mathcal{A}_{\text{even}}$

and $\mathcal{A}_{\text{odd}}$ is ι-antiinvariant.

This completes the proof of Lemma 4.2.3. □

$\mathbf{Proj}_S(\mathcal{A})$ is a fibration of S by rational surfaces. Let $\pi_{\mathcal{A}} \colon \mathbf{Proj}_S(\mathcal{A}) \to S$ denote the natural projection map. The inclusion $\mathcal{A} \subset \mathcal{R}(X, \mathcal{L})$ yields a factorisation of $\pi \colon X \to S$ as

$$X \overset{\phi}{\dashrightarrow} X^c = \mathbf{Proj}_S\big(\mathcal{R}(X,\mathcal{L})\big) \xrightarrow{\psi} \mathbf{Proj}_S(\mathcal{A}) \xrightarrow{\pi_{\mathcal{A}}} S.$$

We will attempt to construct $\mathcal{A}$ first, then use the properties of the map ψ to reconstruct $\mathcal{R}(X, \mathcal{L})$.

As $\mathcal{A}$ is generated by $\mathcal{E}_1$ and $\mathcal{E}_2$, we might expect that $\mathcal{A}$ can be reconstructed from the locally free sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ and the map σ_2 that relates them. The next proposition, our analogue of [CP06, Lemma 4.4], gives us a way to do this:

Proposition 4.2.4. *With notation as above, there are exact sequences*

$$(*) \qquad \mathrm{Sym}^2(\mathcal{E}_1 \wedge \mathcal{E}_1) \otimes \mathrm{Sym}^{n-2}(\mathcal{E}_2) \xrightarrow{i_n} \mathrm{Sym}^n(\mathcal{E}_2) \longrightarrow \mathcal{A}_{2n} \longrightarrow 0 \quad (n \geq 2)$$
$$(**) \qquad \mathcal{E}_1 \otimes (\mathcal{E}_1 \wedge \mathcal{E}_1) \otimes \mathcal{A}_{2n-2} \xrightarrow{j_n} \mathcal{E}_1 \otimes \mathcal{A}_{2n} \longrightarrow \mathcal{A}_{2n+1} \longrightarrow 0 \qquad (n \geq 1)$$

where

$$i_n\big((x_i \wedge x_j)(x_k \wedge x_l) \otimes r\big) := \big(\sigma_2(x_i x_k)\sigma_2(x_j x_l) - \sigma_2(x_i x_l)\sigma_2(x_j x_k)\big)r,$$
$$j_n\big(l \otimes (x_i \wedge x_j) \otimes r\big) := x_i \otimes \big(\sigma_2(x_j l) r\big) - x_j \otimes \big(\sigma_2(x_i l) r\big).$$

Furthermore, if $n = 2$ then sequence $(*)$ is also exact on the left.

Proof. (Based upon the proof of [CP06, Lemma 4.4]) The maps $\mathrm{Sym}^n(\mathcal{E}_2) \to \mathcal{A}_{2n}$ and $\mathcal{E}_1 \otimes \mathcal{A}_{2n} \to \mathcal{A}_{2n+1}$, induced by the ring structure of $\mathcal{A}$, are surjective because $\mathcal{A}$ is generated in degree ≤ 2 by definition. Since $\mathcal{E}_n$ and $\mathcal{A}_n$ are locally free, the respective kernels are locally free also. Furthermore, both sequences are complexes, by virtue of associativity and commutativity in $\mathcal{R}(X, \mathcal{L})$.

It remains to show that $(*)$ and $(**)$ are exact in the middle. Since the kernels of the maps to $\mathcal{A}_n$ are locally free, it is enough to prove this on the fibres of the associated vector bundles.

We begin with sequence $(*)$. Suppose that f is contained in the kernel of the map to $\mathcal{A}_{2n}$. We wish to show that f is also in the image of i_n.

If the fibre of $\pi\colon X \to S$ over the point under consideration is hyperelliptic, then $\mathcal{E}_2$ is generated by the images $\sigma_2(x_i x_j)$ for all $i, j \in \{1, 2, 3\}$. Express f in terms of these generators. Then perform the following algorithm on f:

(i) If any monomial of f contains a factor of $\sigma_2(x_1x_i)\sigma_2(x_1x_j)$, with $i, j \in \{2, 3\}$, replace this factor with $\sigma_2(x_1^2)\sigma_2(x_ix_j)$.

(ii) Repeat step (i) until it terminates.

(iii) If any monomial of f contains a factor of $\sigma_2(x_1x_3)\sigma_2(x_2x_i)$, with $i \in \{2, 3\}$, replace this factor with $\sigma_2(x_1x_2)\sigma_2(x_3x_i)$.

(iv) If any monomial of f contains a factor of $\sigma_2(x_2x_3)\sigma_2(x_2x_3)$, replace this factor with $\sigma_2(x_2^2)\sigma_2(x_3^2)$.

(v) Repeat step (iv) until it terminates.

(vi) Collect like terms in f and simplify.

Call the result f'. Note that the kernel of the map to $\mathcal{A}_{2n}$ is closed under these operations, so f' is in this kernel. Furthermore, $\mathrm{im}(i_n)$ is also closed under these operations and their inverses, so $f \in \mathrm{im}(i_n)$ if and only if $f' \in \mathrm{im}(i_n)$.

Now, any monomial in f' must have the form

$$\sigma_2(x_1^2)^{n_{1,1}}\sigma_2(x_1x_2)^{n_{1,2}}\sigma_2(x_2^2)^{n_{2,2}}\sigma_2(x_2x_3)^{n_{2,3}}\sigma_2(x_3^2)^{n_{3,3}}\sigma_2(x_1x_3)^{n_{1,3}},$$

with $n_{1,2}, n_{2,3}, n_{1,3} \in \{0, 1\}$ and $n_{1,3} = 1$ only if $n_{1,2} = n_{2,2} = n_{2,3} = 0$. However, under the map to $\mathcal{A}_{2n}$ there are no relations between monomials of this form so, since f' is in the kernel of this map, f' must be the zero polynomial. But $0 \in \mathrm{im}(i_n)$, so $f \in \mathrm{im}(i_n)$ also.

The proof for points corresponding to unigonal fibres is very similar. This time, $\mathcal{E}_2$ is generated by y and the images $\sigma_2(x_i x_j)$ for all $i, j \in \{1, 2, 3\}$, with $(i, j) \neq (1, 1)$. We perform the same set of operations on f, but with step (i) replaced by

(i') If any monomial of f contains a factor of $\sigma_2(x_1 x_i)\sigma_2(x_1 x_j)$, with $i, j \in \{2, 3\}$, replace this factor with $(a\sigma_2(x_2^2) + b\sigma_2(x_2 x_3))\sigma_2(x_i x_j)$, where the degree 2 relation in the unigonal fibre is given by $q(x_1, x_2, x_3) = x_1^2 - x_2(ax_2 + bx_3) = 0$ for some $a, b \in \mathbb{C}$.

Any monomial in the resulting f' must have the form

$$y^{n_0}\sigma_2(x_1x_2)^{n_{1,2}}\sigma_2(x_2^2)^{n_{2,2}}\sigma_2(x_2x_3)^{n_{2,3}}\sigma_2(x_3^2)^{n_{3,3}}\sigma_2(x_1x_3)^{n_{1,3}},$$

with $n_{1,2}, n_{2,3}, n_{1,3} \in \{0, 1\}$ and $n_{1,3} = 1$ only if $n_{1,2} = n_{2,2} = n_{2,3} = 0$. With this, the remainder of the proof proceeds exactly as in the hyperelliptic case.

It remains to show that this sequence is exact on the left when $n = 2$. This will again follow from the corresponding statement on the fibres of the associated vector bundles. As the map induced by i_2 on the fibres of the associated vector bundles is linear, in order to prove that it is injective we need only show that the dimension (as a complex vector space) of its domain is equal to that of its image. A simple calculation yields that the dimension of a fibre of $\mathcal{A}_4$ is 21, and the dimension of a fibre of $\mathrm{Sym}^2(\mathcal{E}_2)$ is 15. So, as sequence $(*)$ is exact in the middle, the image of i_2 has dimension 6. But a fibre of $\mathrm{Sym}^2(\mathcal{E}_1 \wedge \mathcal{E}_1)$ also has dimension 6. Hence, i_2 is injective and sequence (1) is exact on the left when $n = 2$.

Next we consider sequence $(**)$. Given f contained in the kernel of the map to $\mathcal{A}_{2n+1}$, we wish to show that f is contained in the image of j_n.

First consider the case where the fibre of $\pi\colon X \to S$ over the point under consideration is hyperelliptic. Then the fibre of the $\mathcal{O}_S$-algebra $\mathcal{A}$ over this point is isomorphic to $\mathbb{C}[x_1, x_2, x_3]$. Since the x_i form a basis for the fibre of $\mathcal{E}_1$, we may write f as

$$f = x_1 \otimes f_1 + x_2 \otimes f_2 + x_3 \otimes f_3$$

for some $f_1, f_2, f_3 \in \mathbb{C}[x_1, x_2, x_3]$ of degree $2n$. This maps to $x_1 f_1 + x_2 f_2 + x_3 f_3$ under the map to $\mathcal{A}_{2n+1}$, so the condition that f is in the kernel of this map is equivalent to

$$x_1 f_1 + x_2 f_2 + x_3 f_3 = 0.$$

Using this equation, we have $x_1 | (x_2 f_2 + x_3 f_3)$. This implies that f_2 and f_3 have the form

$$\begin{aligned} f_2 &= x_1\, r_2(x_1, x_2, x_3) + x_3\, s_{23}(x_2, x_3), \\ f_3 &= x_1\, r_3(x_1, x_2, x_3) - x_2\, s_{23}(x_2, x_3), \end{aligned}$$

for $r_i, s_{ij} \in \mathbb{C}[x_1, x_2, x_3]$ of degree $(2n-1)$. Repeating this process for x_2 and x_3, we get

$$\begin{aligned} f_1 &= x_2 x_3\, r_1(x_1, x_2, x_3) + x_2\, s_{12}(x_1, x_2) + x_3\, s_{13}(x_1, x_3), \\ f_2 &= x_1 x_3\, r_2(x_1, x_2, x_3) - x_1\, s_{12}(x_1, x_2) + x_3\, s_{23}(x_2, x_3), \\ f_3 &= x_1 x_2\, r_3(x_1, x_2, x_3) - x_1\, s_{13}(x_1, x_3) - x_2\, s_{23}(x_2, x_3), \end{aligned}$$

for $r_i, s_{ij} \in \mathbb{C}[x_1, x_2, x_3]$ of degrees $(2n-2)$ and $(2n-1)$ respectively. Furthermore, as f is in the kernel of the map to $\mathcal{A}_{2n+1}$, we must have $r_1 + r_2 + r_3 = 0$.

Let $l_{ij}(x_i, x_j)$ be any linear factor of $s_{ij}(x_i, x_j)$ and write

$$s_{ij}(x_i, x_j) = l_{ij}(x_i, x_j)\, s'_{ij}(x_i, x_j)$$

for $s'_{ij} \in \mathbb{C}[x_1, x_2, x_3]$ of degree $(2n-2)$. Then we have

$$\begin{aligned}
f = & \, x_1 \otimes (x_2 x_3 \, r_1 + x_2 \, l_{12} \, s'_{12} + x_3 \, l_{13} \, s'_{13}) + \\
& x_2 \otimes (x_1 x_3 \, r_2 - x_1 \, l_{12} \, s'_{12} + x_3 \, l_{23} \, s'_{23}) + \\
& x_3 \otimes (-x_1 x_2 \, r_1 - x_1 x_2 \, r_2 - x_1 \, l_{13} \, s'_{13} - x_2 \, l_{23} \, s'_{23}) \\
= & \, j_n \big(x_2 \otimes (x_1 \wedge x_3) \otimes r_1 + x_1 \otimes (x_2 \wedge x_3) \otimes r_2 + l_{12} \otimes (x_1 \wedge x_2) \otimes s'_{12} + \\
& \quad l_{13} \otimes (x_1 \wedge x_3) \otimes s'_{13} + l_{23} \otimes (x_2 \wedge x_3) \otimes s'_{23} \big).
\end{aligned}$$

Hence $f \in \mathrm{im}(j_n)$ and sequence $(**)$ is exact in the middle.

Finally, we have to show that sequence $(**)$ is exact in the middle when the fibre of $\pi \colon X \to S$ over the point under consideration is unigonal. In this case, the fibre of the $\mathcal{O}_S$-algebra $\mathcal{A}$ over this point is isomorphic to

$$\frac{\mathbb{C}[x_1, x_2, x_3, y]}{(x_1^2 - x_2(ax_2 + bx_3))} = \left(\frac{\mathbb{C}[x_1, x_2, x_3]}{(x_1^2 - x_2(ax_2 + bx_3))} \right) [y]$$

for some $a, b \in \mathbb{C}$.

Once again, let f denote an element of the kernel of the map to $\mathcal{A}_{2n+1}$. Then, since the x_i form a basis for $\mathcal{E}_i$, we may write

$$f = x_1 \otimes f_1 + x_2 \otimes f_2 + x_3 \otimes f_3.$$

Furthermore, using the above characterisation of the fibres of $\mathcal{A}$, without loss of generality we can replace f_1, f_2 and f_3 with their coefficients in $\mathbb{C}[x_1, x_2, x_3]/(x_1^2 - x_2(ax_2 + bx_3))$.

In this ring, we can find uniquely determined expressions

$$\begin{aligned}
f_1 &= x_1 \, g_1(x_2, x_3) + h_1(x_2, x_3), \\
f_2 &= x_1 \, g_2(x_2, x_3) + h_2(x_2, x_3),
\end{aligned}$$

$$f_3 = x_1\, g_3(x_2, x_3) + h_3(x_2, x_3),$$

for $g_i, h_i \in \mathbb{C}[x_2, x_3]$ of degrees $(2n-1)$ and $2n$ respectively. With these expressions, the condition that f be in the kernel of the map to $\mathcal{A}_{2n+1}$ is equivalent to

$$x_1(h_1 + x_2\, g_2 + x_3\, g_3) + x_2(ax_2 + bx_3)\, g_1 + x_2\, h_2 + x_3\, h_3 = 0,$$

which occurs if and only if the following equalities hold:

$$x_2(ax_2 + bx_3)\, g_1 + x_2\, h_2 + x_3\, h_3 = 0 \tag{4.1}$$

$$h_1 + x_2\, g_2 + x_3\, g_3 = 0 \tag{4.2}$$

Equation (4.2) can be analysed by a method similar to that used in the hyperelliptic case to obtain

$$\begin{aligned} h_1 &= x_2 x_3\, r_{h_1}(x_2, x_3) + \alpha\, x_2^{2n} + \beta\, x_3^{2n}, \\ g_2 &= x_3\, r_{g_2}(x_2, x_3) - \alpha\, x_2^{2n-1}, \\ g_3 &= x_2\, r_{g_3}(x_2, x_3) - \beta\, x_3^{2n-1}, \end{aligned}$$

for $r_{g_i}, r_{h_i} \in \mathbb{C}[x_2, x_3]$ of degree $(2n-2)$ and $\alpha, \beta \in \mathbb{C}$. Furthermore, by substituting back into equation (4.2) we must have that $r_{h_1} + r_{g_2} + r_{g_3} = 0$.

Equation (4.1) is slightly more problematic. From this equation, we must have $x_2 | x_3 h_3$ and $x_3 | (ax_2^2\, g_1 + x_2\, h_2)$. This means that g_1, h_2 and h_3 have the form

$$\begin{aligned} g_1 &= x_3\, r_{g_1}(x_2, x_3) + \gamma\, x_2^{2n-1}, \\ h_2 &= x_3\, r_{h_2}(x_2, x_3) - a\gamma\, x_2^{2n}, \\ h_3 &= x_2\, r_{h_3}(x_2, x_3), \end{aligned}$$

for $r_{g_i}, r_{h_i} \in \mathbb{C}[x_2, x_3]$ of degrees $(2n-2)$ and $(2n-1)$ respectively, and $\gamma \in \mathbb{C}$. By substituting back into equation (4.1) we obtain $(ax_2+bx_3)\, r_{g_1}+r_{h_2}+r_{h_3}+b\gamma\, x_2^{2n-1}=0$.

Finally, let $l_{h_2}(x_2, x_3)$ be a linear factor of $r_{h_2}(x_2, x_3)$, so that

$$r_{h_2}(x_2, x_3) = l_{h_2}(x_2, x_3)\, r'_{h_2}(x_2, x_3),$$

for some $r'_{h_2} \in \mathbb{C}[x_2, x_3]$ of degree $(2n-2)$.

Putting all of this together, we get that

$$\begin{aligned} f = {} & x_1 \otimes \big(x_1(x_3\, r_{g_1} + \gamma\, x_2^{2n-1}) + x_2x_3\, r_{h_1} + \alpha\, x_2^{2n} + \beta\, x_3^{2n}\big) + \\ & x_2 \otimes \big(x_1(x_3\, r_{g_2} - \alpha\, x_2^{2n-1}) + x_3\, l_{h_2}\, r'_{h_2} - \gamma(ax_2+bx_3)x_2^{2n-1} + b\gamma\, x_3x_2^{2n-1}\big) + \\ & x_3 \otimes \big(x_1(-x_2\, r_{g_2} - x_2\, r_{h_1} - \beta\, x_3^{2n-1}) - x_2(ax_2+bx_3)\, r_{g_1} - x_2\, l_{h_2}\, r'_{h_2} - b\gamma\, x_2^{2n}\big) \\ = {} & j_n\big(x_1 \otimes (x_1 \wedge x_3) \otimes r_{g_1} + x_1 \otimes (x_1 \wedge x_2) \otimes \gamma\, x_2^{2n-2} + x_2 \otimes (x_1 \wedge x_3) \otimes r_{h_1} + \\ & x_2 \otimes (x_1 \wedge x_2) \otimes \alpha\, x_2^{2n-2} + x_3 \otimes (x_1 \wedge x_3) \otimes \beta\, x_3^{2n-2} + x_1 \otimes (x_2 \wedge x_3) \otimes r_{g_2} + \\ & l_{h_2} \otimes (x_2 \wedge x_3) \otimes r'_{h_2} + x_2 \otimes (x_2 \wedge x_3) \otimes b\gamma\, x_2^{2n-2}\big) \end{aligned}$$

Thus, f is in the image of j_n and sequence $(**)$ is exact in the middle. This completes the proof of Proposition 4.2.4. □

The exact sequences $(*)$ and $(**)$ in Proposition 4.2.4 allow us to describe $\mathcal{A}_{\text{even}}$ as a quotient algebra of $\operatorname{Sym}(\mathcal{E}_2)$ and $\mathcal{A}_{\text{odd}}$ as an $\mathcal{A}_{\text{even}}$-module. The multiplication map $\mathcal{A}_{\text{odd}} \times \mathcal{A}_{\text{odd}} \to \mathcal{A}_{\text{even}}$ is induced by the composition

$$\mathcal{E}_1 \otimes \mathcal{E}_1 \xrightarrow{\mu_{1,1}} \operatorname{Sym}^2(\mathcal{E}_1) \xrightarrow{\sigma_2} \mathcal{E}_2.$$

Thus, $\mathcal{A}$ is completely determined as an $\mathcal{O}_S$-algebra by the locally free sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ and the map $\sigma_2 \colon \operatorname{Sym}^2(\mathcal{E}_1) \to \mathcal{E}_2$.

The structure of $\mathcal{A}_{\text{even}}$ as a quotient algebra of $\operatorname{Sym}(\mathcal{E}_2)$ gives a Veronese embedding of $\mathbf{Proj}_S(\mathcal{A})$ into $\mathbb{P}_S(\mathcal{E}_2)$ that commutes with the projection to S. The projective space bundle $\mathbb{P}_S(\mathcal{E}_2)$ comes equipped with natural invertible sheaves $\mathcal{O}(n)$ for all $n \in \mathbb{Z}$, which induce invertible sheaves $\mathcal{O}_{\mathbf{Proj}_S(\mathcal{A})}(2n)$ on $\mathbf{Proj}_S(\mathcal{A})$.

Now that we have a way to construct $\mathcal{A}$, we would like to find a way to reconstruct $\mathcal{R}(X, \mathcal{L})$ from it. By Lemma 4.2.3, we can already construct $\mathcal{R}(X, \mathcal{L})$ as an $\mathcal{A}$-module. However, we need to give $\mathcal{R}(X, \mathcal{L})$ a multiplicative structure to make it into an $\mathcal{A}$-algebra. In order to do this, we need to determine the multiplication map from $\mathcal{E}_3^+ \otimes \mathcal{E}_3^+$ to $\mathcal{E}_6$. By Lemma 4.2.3, this multiplication map has image contained in $\mathcal{A}_6$. So the ring structure on $\mathcal{R}(X, \mathcal{L})$ induces a map

$$\beta \colon (\mathcal{E}_3^+)^2 \longrightarrow \mathcal{A}_6.$$

To determine β, we will study the map $\psi \colon X^c \to \mathbf{Proj}_S(\mathcal{A})$. First, however, we need a definition.

Definition 4.2.5. *Let P be a point in the support of τ. The fibre of $\mathbf{Proj}_S(\mathcal{A})$ over P is of the form*

$$\{x_1^2 - x_2(ax_2 + bx_3) = 0\} \subset \mathbb{P}_{(1,1,1,2)}[x_1, x_2, x_3, y].$$

This is a quadric cone and is singular at the point $(0{:}0{:}0{:}1)$.

Taking all such singular points associated to the points of $\mathrm{Supp}(\tau)$*, we get a subset of $\mathbf{Proj}_S(\mathcal{A})$ that we will denote by $\mathcal{P}$. Note that the projection onto S maps $\mathcal{P}$ bijectively onto* $\mathrm{Supp}(\tau)$*.*

Then we have the following analogue of [CP06, Theorem 4.7]:

Proposition 4.2.6. *$X^c = \mathbf{Proj}_S(\mathcal{R}(X, \mathcal{L}))$ is a double cover of $\mathbf{Proj}_S(\mathcal{A})$, with branch locus consisting of the set of isolated points $\mathcal{P}$ together with the divisor $B_{\mathcal{A}}$ in the linear system $|\mathcal{O}_{\mathbf{Proj}_S(\mathcal{A})}(6) \otimes \pi_{\mathcal{A}}^*(\mathcal{E}_3^+)^{-2}|$ determined by β ($B_{\mathcal{A}}$ is thus disjoint from $\mathcal{P}$).*

Proof. (Following the proof of [CP06, Theorem 4.7]) Note first that $\psi \colon X^c \to \mathbf{Proj}_S(\mathcal{A})$ is a double cover by Lemma 4.2.3. It just remains to calculate the branch locus of ψ.

Since the question is local on S, we may use the same method as in the proof of Lemma 4.2.3 and restrict to an affine open set S_0 over which X^c is isomorphic to the subscheme of $\mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z] \times S_0$ defined by the equations

$$f_2(x_1, x_2, x_3, y, t) = 0, \qquad z^2 = f_6(x_1, x_2, x_3, y, t),$$

where t is a parameter on S_0. Furthermore, we note that if $x_1 = x_2 = x_3 = y = 0$ then $z = 0$, which is impossible, so the x_i's and y cannot simultaneously vanish.

At a point where $x_i \neq 0$ for some i, we can localise both equations by dividing by x_i^2, respectively by x_i^6. Then $z = 0$ is the ramification divisor and $f_6 = 0$ is the branch locus. This equation defines exactly the divisor $B_{\mathcal{A}} \subset \mathbf{Proj}_S(\mathcal{A})$.

At a point where $x_1 = x_2 = x_3 = 0$, we may assume that $y = 1$ and we have a point of $\mathcal{P}$. Note that, since the points $(0:0:0:1:a)$ and $(0:0:0:1:-a)$ are identified in $\mathbb{P}(1,1,1,2,3)$ for any $a \in \mathbb{C}$, this point must be a branch point of ψ. Furthermore, by Theorem 3.2.2, f_6 cannot vanish at such a point, so $B_{\mathcal{A}}$ is disjoint from $\mathcal{P}$. $\square$

Putting the results of this section together, we can list the data required to construct the relative log canonical model of a semistable terminal threefold fibred by K3 surfaces of degree two.

Definition 4.2.7. *Let $(X, \pi, \mathcal{L})$ be a semistable terminal threefold fibred by K3 surfaces of degree two over a nonsingular curve S and suppose further that $\mathcal{L}$ is locally π-flat. Then define the* associated 5-tuple *of the pair $(X, \mathcal{L})$ over S, denoted $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$, as follows:*

- $\mathcal{E}_1 = \pi_*(\omega_X \otimes \mathcal{L})$.
- *τ is the effective divisor on S whose structure sheaf is isomorphic to $\mathcal{T}_2$.*

- $\xi \in \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_\tau, \mathrm{Sym}^2(\mathcal{E}_1))/\mathrm{Aut}_{\mathcal{O}_s}(\mathcal{O}_\tau)$ *is the isomorphism class corresponding to the pair* $(\mathcal{E}_2, \sigma_2)$ *in the sequence*

$$0 \longrightarrow \mathrm{Sym}^2(\mathcal{E}_1) \xrightarrow{\sigma_2} \mathcal{E}_2 \longrightarrow \mathcal{O}_\tau \longrightarrow 0.$$

- $\mathcal{E}_3^+$ *is the* ι*-invariant part of* $\pi_*(\omega_X^3 \otimes \mathcal{L}^3)$.
- $\beta \in \mathbb{P}(H^0(S, \mathcal{A}_6 \otimes (\mathcal{E}_3^+)^{-2})) \cong |\mathcal{O}_{\mathbf{Proj}_S(\mathcal{A})}(6) \otimes \pi_{\mathcal{A}}^*(\mathcal{E}_3^+)^{-2}|$ *is the class of a section with associated divisor* $B_{\mathcal{A}}$.

Remark 4.2.8. We need one more piece of data than Catanese and Pignatelli [CP06]: the line bundle $\mathcal{E}_3^+$. This is because they construct the relative log canonical model of the pair $(X, 0)$, but we are constructing the relative log canonical model of the pair $(X, \mathcal{L})$. The extra data in our case is needed to determine the polarisation divisor $\mathcal{L}$.

4.3 A Generality Result

In this section we will give a method, based upon the results of Section 4.2, to construct relative log canonical models of semistable terminal threefolds fibred by K3 surfaces of degree two, and prove a result about the generality of this construction.

Fix a nonsingular complex curve S. We begin with a 5-tuple of data $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ on S, defined by:

- $\mathcal{E}_1$ is a rank 3 vector bundle on S.
- τ is an effective divisor on S.
- $\xi \in \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_\tau, \mathrm{Sym}^2(\mathcal{E}_1))/\mathrm{Aut}_{\mathcal{O}_s}(\mathcal{O}_\tau)$ yields a vector bundle $\mathcal{E}_2$ on S and a map $\sigma_2\colon \mathrm{Sym}^2(\mathcal{E}_1) \to \mathcal{E}_2$.
- $\mathcal{E}_3^+$ is a line bundle on S.

- $\beta \in \mathbb{P}(H^0(S, \mathcal{A}_6 \otimes (\mathcal{E}_3^+)^{-2}))$, where $\mathcal{A}_6$ is defined using $\mathcal{E}_1$, $\mathcal{E}_2$, σ_2 and the exact sequences of Proposition 4.2.4.

Given this data, we begin by constructing a sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ using the exact sequences of Proposition 4.2.4. Then we may define a sheaf of $\mathcal{O}_S$-algebras

$$\mathcal{R} := \mathcal{A} \oplus (\mathcal{A}[-3] \otimes \mathcal{E}_3^+),$$

with multiplicative structure induced by $\mathcal{A}$ and the map $(\mathcal{E}_3^+)^2 \to \mathcal{A}_6$ defined by β.

Definition 4.3.1. *We say that a 5-tuple $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ is* admissible *if the sheaf of algebras $\mathcal{R}$ constructed from it satisfies the following conditions:*

(i) *Let $B_{\mathcal{A}}$ be the divisor of β on $\mathbf{Proj}_S(\mathcal{A})$; then $B_{\mathcal{A}}$ does not contain any point of the set $\mathcal{P}$ defined in 4.2.5.*

(ii) *$\mathbf{Proj}_S(\mathcal{R})$ has at worst canonical singularities.*

(iii) *There exists an analytic resolution $f\colon Y \to \mathbf{Proj}_S(\mathcal{R})$ that modifies only finitely many fibres of $\pi_{\mathcal{R}}\colon \mathbf{Proj}_S(\mathcal{R}) \to S$, such that all fibres of $\pi_{\mathcal{R}} \circ f$ are semistable.*

We have the following generality result, analogue of [CP06, Theorem 4.12]:

Theorem 4.3.2. *Fix a nonsingular complex curve S. Let $(X, \pi, \mathcal{L})$ be a semistable terminal threefold fibred by K3 surfaces of degree two and suppose further that $\mathcal{L}$ is locally π-flat. Then the associated 5-tuple of the pair $(X, \mathcal{L})$ is admissible.*

Conversely, let $\mathcal{R}$ be a sheaf of $\mathcal{O}_S$-algebras defined by an admissible 5-tuple of data $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ on S. Then there exists a semistable terminal threefold fibred by K3 surfaces of degree two $(X, \pi, \mathcal{L})$ with $\mathcal{L}$ locally π-flat, such that $\mathbf{Proj}_S(\mathcal{R})$ is the relative log canonical model of the pair $(X, \mathcal{L})$ and $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ is its associated 5-tuple.

Remark 4.3.3. This theorem provides a bijection between admissible 5-tuples and relative log canonical models of semistable terminal threefolds fibred by K3 surfaces of degree two. Note, however, that there may be many birationally equivalent semistable terminal threefolds fibred by K3 surfaces of degree two that all have the same relative log canonical model, and so determine the same associated 5-tuple.

This is closely related to the so-called "flop problem", i.e. the fact that in dimension ≥ 3 minimal models are not unique, but merely isomorphic in codimension one. The flop problem means that, whilst a given threefold determines a unique canonical model, it may have many different minimal models. In our case, the minimal models correspond to different semistable terminal threefolds fibred by K3 surfaces of degree two that determine the same relative log canonical model. Further details may be found in [KM98, Section 3.8].

Remark 4.3.4. The fact that the singularities appearing in the relative log canonical model X^c of $(X, \mathcal{L})$ are at worst canonical fits nicely with the fact (Corollary 3.2.4) that the singularities occurring in the fibres of X^c are at worst semi log canonical. The relationship between these two types of singularities is given by Kollár and Shepherd-Barron [KSB88, Theorem 5.1], who show that if $X \to \Delta$ is a Gorenstein one-parameter deformation of a surface X_0 that admits a semistable resolution, then X has canonical singularities if and only if X_0 has semi log canonical surface singularities and the general fibre has at worst rational double points.

Proof. We begin by letting $(X, \pi, \mathcal{L})$ be a semistable terminal threefold fibred by K3 surfaces of degree two over S, such that $\mathcal{L}$ is locally π-flat. We want to show that the associated 5-tuple of $(X, \mathcal{L})$ is admissible. Note that condition (i) in the definition of admissible follows immediately from Proposition 4.2.6.

To show that the remaining two conditions hold, we will use the locally Kulikov model constructed in Section 2.4. Recall that given $\pi\colon X \to S$ as above, we may find $\pi''\colon X'' \to S$ birational to X over S and a locally π''-flat line bundle $\mathcal{L}''$ on X'', such that $(X, \mathcal{L})$ and $(X'', \mathcal{L}'')$ define the same relative log canonical algebra over S, and hence the same associated 5-tuple. Furthermore, the natural map $\phi''\colon X'' \to X^c$ from X'' to the relative log canonical model X^c is a morphism. As X'' is nonsingular and semistable, and ϕ'' is an isomorphism outside of finitely may fibres, this proves condition (iii).

It remains to show that X^c has at worst canonical singularities. In order to do this, it is enough to look locally around any fibre of π''. So let $s \in S$ be any closed point, and let U_s be an open neighbourhood of s. Define $X''_{U_s} := \pi''^{-1}(U_s)$. By the local π''-flatness property, we can find a divisor H_s that is flat over U_s in the linear system defined on X''_{U_s} by $\mathcal{L}''$. Furthermore, as U_s is open, the relative log canonical model $X''^c_{U_s}$ of the pair $(X''_{U_s}, \mathcal{O}_{X''_{U_s}}(H_s))$ agrees with the relative log canonical model of $(X'', \mathcal{L}'')$ over U_s. So it suffices to show that $X''^c_{U_s}$ has at worst canonical singularities for all s.

In order to do this, note first that the intersection of H_s with a general fibre is irreducible so, since H_s is flat over U_s, we must have H_s irreducible. Thus, as X''_{U_s} is nonsingular, by Proposition 2.1.5, $\operatorname{discrep}(X''_{U_s}, H_s) = 0$. So, as no components of H are contracted by ϕ'', by [KM98, Proposition 3.51] we see that $\operatorname{discrep}(X''^c_{U_s}, \phi''_+ H_s) \geq 0$ and the log pair $(X''^c_{U_s}, \phi''_+ H_s)$ is canonical. But $\phi''_+ H_s$ is effective on $X''^c_{U_s}$ so, by [KM98, Corollary 2.35], the log pair $(X''^c_{U_s}, 0)$ is also canonical and $X''^c_{U_s}$ has at worst canonical singularities. This proves condition (ii).

Next we prove the converse statement. Let $\mathcal{R}$ be a sheaf of $\mathcal{O}_S$-algebras defined by an admissible 5-tuple $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$. Define $Y := \mathbf{Proj}_S(\mathcal{R})$ and let $\pi_Y \colon Y \to S$ denote the natural projection. As in the proof of Lemma 4.2.3, over an affine open set $U \subset S$ we can view Y as a subvariety in $\mathbb{P}(1,1,1,2,3) \times S_0$. Furthermore, by the local description in Lemma 4.2.1 we see that Y does not intersect the singular locus and that $\omega_Y|_{\pi_Y^{-1}(U)}$ is trivial. Therefore Y is Gorenstein and over each such open set the sheaf $\mathcal{O}(1)$ induced on Y by the weighted projective space structure is invertible. These invertible sheaves glue to give a invertible sheaf $\mathcal{O}_Y(1)$ on Y. Thus, the sheaf $\mathcal{O}_Y(1) \otimes \omega_Y^{-1}$ is invertible and, by the local description of Y in Lemma 4.2.1, this sheaf is locally π_Y-flat and induces an ample invertible sheaf with self-intersection number two on a general fibre of π_Y.

By condition (iii) in the definition of an admissible 5-tuple, we may find a semistable (analytic) resolution $f\colon \overline{Y} \to Y$ that is an isomorphism outside of finitely many fibres. Let $\overline{\pi} := \pi_Y \circ f$. Then, we have:

Lemma 4.3.5. *With notation as above, we may find a locally $\overline{\pi}$-flat line bundle $\overline{\mathcal{L}}$ on $\overline{Y}$ making $(\overline{Y}, \overline{\pi}, \overline{\mathcal{L}})$ into a semistable analytic threefold fibred by K3 surfaces of degree two over S. Furthermore, $\overline{\mathcal{L}}$ may be chosen such that the relative log canonical algebra $\mathcal{R}(\overline{Y}, \overline{\mathcal{L}})$ is equal to $\mathcal{R}$.*

Proof. We begin by defining the polarisation $\overline{\mathcal{L}}$. Choose an ample invertible sheaf $\mathcal{M}$ on S. Then for some $m > 0$, the sheaf $(\pi_Y)_*(\mathcal{O}_Y(1) \otimes \omega_Y^{-1}) \otimes \mathcal{M}^m$ is generated by its global sections. Let H denote a general member of the linear system $|\mathcal{O}_Y(1) \otimes \omega_Y^{-1} \otimes \pi_Y^* \mathcal{M}^m|$ and define $\overline{\mathcal{L}} := \mathcal{O}_{\overline{Y}}(f_+^{-1} H) \otimes \overline{\pi}^* \mathcal{M}^{-m}$. Then, as f is an isomorphism outside of finitely many fibres, $(\overline{Y}, \overline{\pi}, \overline{\mathcal{L}})$ has the structure of a semistable analytic threefold fibred by K3 surfaces of degree two over S. It just remains to show that $\overline{\mathcal{L}}$ is locally $\overline{\pi}$-flat and $\mathcal{R}$ is the relative log canonical algebra of $(\overline{Y}, \overline{\mathcal{L}})$.

In order to do this, we begin by showing that H may be chosen to avoid the worst singularities of Y. Specifically, we want to avoid singularities that are not *compound Du Val*:

Definition 4.3.6. [KM98, 5.32] *Let $0 \in Y$ be a threefold singularity. We say that it is a* compound Du Val *singularity if a general hypersurface section $0 \in H \subset Y$ is a rational double point (i.e. a Du Val singularity).*

We start by examining the linear system $|\mathcal{O}_Y(1) \otimes \omega_Y^{-1} \otimes \pi_Y^* \mathcal{M}^m|$ in which H moves. Note that as $(\pi_Y)_*(\mathcal{O}_Y(1) \otimes \omega_Y^{-1}) \otimes \mathcal{M}^m$ is generated by its global sections, for any affine open set $U \subset S$ the sections in $H^0(Y, \mathcal{O}_Y(1) \otimes \omega_Y^{-1} \otimes \pi_Y^* \mathcal{M}^m)$ generate $H^0(\pi_Y^{-1}(U), \mathcal{O}_Y(1) \otimes \omega_Y^{-1} \otimes \pi_Y^* \mathcal{M}^m)$ as an $\mathcal{O}_{\pi_Y^{-1}(U)}$-module, so we may study this linear system locally over S.

Let $U \subset S$ be an affine open set. Then as in the proof of Lemma 4.2.3, we can view $\pi_Y^{-1}(U)$ as a complete intersection

$$\pi_Y^{-1}(U) \cong \{f_2(t) = f_6(t) = 0\} \subset \mathbb{P}(1,1,1,2,3) \times U.$$

As $\mathcal{M}|_U \cong \mathcal{O}_U$, we have $\pi_Y^*\mathcal{M}^m|_{\pi_Y^{-1}(U)} \cong \mathcal{O}_{\pi_Y^{-1}(U)}$ and, since $\omega_Y|_{\pi_Y^{-1}(U)}$ is trivial, the restriction of $(\mathcal{O}_Y(1) \otimes \omega_Y^{-1} \otimes \pi_Y^*\mathcal{M}^m)$ to $\pi_Y^{-1}(U)$ is just the sheaf induced from $\mathcal{O}(1)$ on $\mathbb{P}(1,1,1,2,3) \times U$. Furthermore, by the local descriptions of $f_2(t)$ and $f_6(t)$ given in Lemma 4.2.1, we see that this sheaf defines a linear system on $\pi_Y^{-1}(U)$ that has no fixed components and is base point free outside of the points over the set $\mathcal{P}$ defined in 4.2.5. Thus, as $H^0(\pi_Y^{-1}(U), \mathcal{O}_Y(1) \otimes \omega_Y^{-1} \otimes \pi_Y^*\mathcal{M}^m)$ is generated as an $\mathcal{O}_{\pi_Y^{-1}(U)}$-module by the sections in $H^0(Y, \mathcal{O}_Y(1) \otimes \omega_Y^{-1} \otimes \pi_Y^*\mathcal{M}^m)$, we see that the linear system $|\mathcal{O}_Y(1) \otimes \omega_Y^{-1} \otimes \pi_Y^*\mathcal{M}^m|$ has no base points or fixed components on Y outside of the points over the set $\mathcal{P}$.

As Y has at worst canonical singularities, by [KM98, Corollary 5.40] all but finitely many of the singular points are compound Du Val. So, apart from the points lying over the set $\mathcal{P}$, we may assume that the only singularities of Y lying on H are compound Du Val. Furthermore, by Bertini's theorem we may assume that H is irreducible and nonsingular outside of the singular points of Y and the points lying over $\mathcal{P}$. In particular H cannot contain any components of fibres, so is horizontal and thus flat over S by the proof of Proposition 2.4.7. This proves the local $\overline{\pi}$-flatness of $\overline{\mathcal{L}}$.

Our next step is to show that, with H chosen as above, the log pair (Y, H) is canonical. This will follow from [KM98, Theorem 5.34] if we can show that all of the singularities in H are rational double points. By the argument above, these singularities arise from compound Du Val points and points lying over $\mathcal{P}$. At a compound Du Val point, the singularity in H is a rational double point by definition. So it just remains to classify the singularities lying over the points of $\mathcal{P}$.

By the proof of Lemma 4.2.1, after a change of coordinates locally we can write $\mathbf{Proj}_S(\mathcal{A})$ as

$$\{f_2(t)=0\} \subset \mathbb{P}_{(1,1,1,2)}[x_1,x_2,x_3,y] \times U,$$

where

$$f_2(t) = x_1^2 - x_2(ax_2+bx_3) + t^r y + t\psi(x_i,t)$$

for some $a, b \in \mathbb{C}$ that are not both zero and t a local parameter on the affine open set $U \subset S$. The weighted projective space structure induces a rank one reflexive sheaf $\mathcal{O}_{\mathbf{Proj}_S(\mathcal{A})}(1)$ locally on $\mathbf{Proj}_S(\mathcal{A})$, a general section of which defines a Weil divisor that has a rational double point singularity of type A_{2r+1} at the point $(0{:}0{:}0{:}1\,;0)$. As $B_{\mathcal{A}}$ does not contain any point of $\mathcal{P}$, around the point $(0:0:0:1\,;0)$ we have that Y is a cyclic double cover of $\mathbf{Proj}_S(\mathcal{A})$ ramified over the point $(0{:}0{:}0{:}1\,;0)$. Thus a divisor defined by a general section of $\mathcal{O}_Y(1)$ is a cyclic double cover of a divisor defined by a general section of $\mathcal{O}_{\mathbf{Proj}_S(\mathcal{A})}(1)$ ramified over the singularity. Therefore, by [KM98, Theorem 5.43], the general section of $\mathcal{O}_Y(1)$ has a rational double point singularity of type A_r.

Thus, we may assume that the only singularities occurring in H are rational double points, so by [KM98, Theorem 5.34] the log pair (Y,H) is canonical. Therefore, by definition we must have

$$\omega_{\overline{Y}} \otimes \mathcal{O}_{\overline{Y}}(f_+^{-1}H) \cong f^*(\omega_Y \otimes \mathcal{O}_Y(H)) \otimes \mathcal{O}_{\overline{Y}}(E),$$

for some effective and f-exceptional E. But by the projection formula this implies that

$$f_*\big(\omega_{\overline{Y}}^n \otimes \mathcal{O}_{\overline{Y}}(f_+^{-1}H)^n\big) \cong \omega_Y^n \otimes \mathcal{O}_Y(nH)$$

for all $n > 0$. Finally, twisting by $\pi_Y^* \mathcal{M}^{-m}$ and noting that $\overline{\pi} = \pi_Y \circ f$, we see that

$$\overline{\pi}_* \big(\omega_{\overline{Y}}^n \otimes \mathcal{O}_{\overline{Y}}(f_+^{-1}H)^n \otimes \overline{\pi}^* \mathcal{M}^{-mn}\big) \cong (\pi_Y)_* \big(\omega_Y^n \otimes \mathcal{O}_Y(nH) \otimes \pi_Y^* \mathcal{M}^{-mn}\big)$$

for all $n > 0$. This gives

$$\overline{\pi}_* \big(\omega_{\overline{Y}}^n \otimes \overline{\mathcal{L}}^n\big) \cong (\pi_Y)_* \mathcal{O}_Y(1)^n$$

for all $n > 0$, so the relative log canonical algebra of $(\overline{Y}, \overline{\mathcal{L}})$ is equal to $\mathcal{R}$, as required. $\square$

We now return to the proof of Theorem 4.3.2. Let $(\overline{Y}, \overline{\pi}, \overline{\mathcal{L}})$ be defined as in Lemma 4.3.5. Then by the same process as was used in Section 2.4 we may find $\pi'' \colon Y'' \to S$ birational to $\overline{Y}$ over S and a locally π''-flat line bundle $\mathcal{L}''$ on Y'' making $(Y'', \pi'', \mathcal{L}'')$ into a semistable analytic threefold fibred by K3 surfaces of degree two, such that Y'' is locally Kulikov, $\mathcal{L}''$ is π''-nef, $\mathcal{R}$ is the relative log canonical algebra of $(Y'', \pi'', \mathcal{L}'')$ and the map $\phi'' \colon Y'' \to Y$ is a birational morphism.

Then, using Theorem 2.4.1, we may find a divisor D supported on the singular fibres of π'' such that $(\mathcal{L}'')^N \otimes \mathcal{O}_{Y''}(D)$ defines a birational morphism

$$\phi^t \colon Y'' \longrightarrow X \subset \mathbb{P}_S\left(\pi''_*\left(\mathcal{L}''^N \otimes \mathcal{O}_{Y''}(D)\right)\right)$$

over S that contracts only finitely many curves, all of which are contained in fibres of π''. Furthermore, examining the proof of this theorem (see [SB83b, Section 2]) we see that any curve C contracted by ϕ^t has $\mathcal{L}''.C = 0$. So the morphism ϕ'' factorises through ϕ^t. We have a diagram:

$$\begin{tikzcd}
\overline{Y} \arrow[r, dashed] \arrow[ddr, "f"'] & Y'' \arrow[dd, "\phi''"] \arrow[dr, "\phi^t"] & \\
& & X \arrow[dl, "\phi"] \\
& Y &
\end{tikzcd}$$

Define $\pi := \pi_Y \circ \phi$. We claim that $\pi \colon X \to S$ is a terminal semistable K3-fibration and that there exists a locally π-flat polarisation $\mathcal{L}$ on X making $(X, \pi, \mathcal{L})$ into a semistable terminal threefold fibred by K3 surfaces of degree two, such that the pair $(X, \mathcal{L})$ has relative log canonical algebra $\mathcal{R}(X, \mathcal{L}) = \mathcal{R}$. As $\mathcal{R}$ uniquely determines the 5-tuple $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$, this will be enough to prove Theorem 4.3.2.

To see that $\pi \colon X \to S$ is a terminal semistable K3-fibration, note that by construction X has Gorenstein terminal singularities, $\phi^t \colon Y'' \to X$ is a small analytic resolution with Y'' semistable and the fibres of π are all reduced divisors with normal crossings outside of the singular points of X.

Now define a sheaf $\mathcal{L} := \phi^t_* \mathcal{L}''$ on X. We claim that $\mathcal{L}$ is invertible. If so, $\mathcal{L}$ is clearly locally π-flat, since $\mathcal{L}''$ is. Furthermore, $(X, \pi, \mathcal{L})$ and $(Y'', \pi'', \mathcal{L}'')$ agree over an open set of S, so $(X, \pi, \mathcal{L})$ is a semistable terminal threefold fibred by K3 surfaces of degree two.

Lemma 4.3.7. *With definitions as above, $\mathcal{L} \cong \phi^*(\mathcal{O}_Y(1) \otimes \omega_Y^{-1})$. In particular, $\mathcal{L}$ is an invertible sheaf on X.*

Proof. In order to prove this we will show that $\mathcal{L}'' \cong (\phi'')^*(\mathcal{O}_Y(1) \otimes \omega_Y^{-1})$; the result then follows by the projection formula and the fact that $\phi'' = \phi \circ \phi^t$.

So consider $\mathcal{L}''$. Define a Cartier divisor H on Y that is flat over S and an invertible sheaf $\mathcal{M}$ on S as in the proof of Lemma 4.3.5. Then, by construction,

$$\mathcal{L}'' \cong \mathcal{O}_{Y''}\left((\phi'')^{-1}_+ H\right) \otimes (\pi'')^* \mathcal{M}^{-m}.$$

Since $\mathcal{O}_Y(1) \otimes \omega_Y^{-1} \cong \mathcal{O}_Y(H) \otimes \pi_Y^* \mathcal{M}^{-m}$, the proof of the lemma will be complete if we can show that $\mathcal{O}_{Y''}((\phi'')^{-1}_+ H) \cong (\phi'')^* \mathcal{O}_Y(H)$. This will follow from the facts that the log pair (Y, H) is canonical and Y'' is locally Kulikov.

Write

$$(\phi'')^* \mathcal{O}_Y(H) \cong \mathcal{O}_{Y''}\left((\phi'')^{-1}_+ H\right) \otimes \mathcal{O}_{Y''}(E),$$

for some effective and ϕ''-exceptional E. Then, since both ω_Y and $\omega_{Y''}$ are trivial in a neighbourhood of any fibre, we must have $\omega_{Y''} \cong (\phi'')^*\omega_Y$ (i.e. ϕ'' is *crepant*). Putting this together we get

$$\omega_{Y''} \otimes \mathcal{O}_{Y''}\big((\phi'')_+^{-1}H\big) \cong (\phi'')^*\big(\omega_Y \otimes \mathcal{O}_Y(H)\big) \otimes \mathcal{O}_{Y''}(-E).$$

But (Y, H) is canonical, so $-E$ must be effective. Hence, $E = 0$ and we have proved that $\mathcal{O}_{Y''}((\phi'')_+^{-1}H) \cong (\phi'')^*\mathcal{O}_Y(H)$, as required. This completes the proof of Lemma 4.3.7. $\square$

Thus, $(X, \pi, \mathcal{L})$ is a semistable terminal threefold fibred by K3 surfaces of degree two and $\mathcal{L}$ is locally π-flat. Finally, from Lemma 4.3.7 we see that $\mathcal{L} \cong \phi^*(\mathcal{O}_Y(1) \otimes \omega_Y^{-1})$ and, since both ω_X and ω_Y are trivial in a neighbourhood of any fibre, we must have $\omega_X \cong \phi^*\omega_Y$. Thus, by the projection formula and the fact that $\pi = \pi_Y \circ \phi$ we see that

$$\pi_*(\omega_X^n \otimes \mathcal{L}^n) \cong (\pi_Y)_*\mathcal{O}_Y(1)^n$$

for all $n \geq 0$, so the relative log canonical algebra of $(X, \mathcal{L})$ is equal to $\mathcal{R}$, as required. This completes the proof of Theorem 4.3.2. $\square$

Chapter 5

Properties of the Constructed Threefolds

5.1 The Canonical Sheaf and the Kodaira Dimension

In this chapter we aim to explicitly calculate the properties of the threefolds constructed in Chapter 4 as the relative log canonical models of threefolds fibred by K3 surfaces of degree two, and of certain resolutions of them.

We begin by setting up some notation. Let S be a nonsingular curve. As in Section 4.3, we begin with an admissible 5-tuple of data $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ on S and use this to construct a sheaf of $\mathcal{O}_S$-algebras $\mathcal{R}$. Define $X := \mathbf{Proj}_S \mathcal{R}$ and let $\pi\colon X \to S$ be the natural projection.

By the proof of Theorem 4.3.2, there are two "nice" resolutions of X. First, there is a semistable terminal threefold fibred by K3 surfaces of degree two $(X^t, \pi^t, \mathcal{L}^t)$ (this is the threefold constructed in the statement of Theorem 4.3.2), such that $\mathcal{L}^t$ is π^t-nef and locally π^t-flat, X is the relative log canonical model of $(X^t, \mathcal{L}^t)$ over S, and the birational map $\phi\colon X^t \to X$ is a crepant morphism (i.e. $\omega_{X^t} \cong \phi^*\omega_X$).

Second, there is a semistable smooth analytic threefold fibred by K3 surfaces of degree two $(Y, \pi_Y, \mathcal{L}_Y)$ (called Y'' in the proof of 4.3.2), such that Y is locally Kulikov, $\mathcal{L}_Y$ is π_Y-nef and locally π_Y-flat, X is the relative log canonical model of $(Y, \mathcal{L}_Y)$ over S, and the birational map $\phi_Y \colon Y \to X$ is a morphism. Furthermore, Y is a small analytic resolution of X^t.

We have a diagram:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\phi_Y"'] \arrow[ddr, bend right, "\pi_Y"'] & & X^t \arrow[dl, "\phi"] \arrow[ddl, bend left, "\pi^t"] \\ & X \arrow[d, "\pi"] & \\ & S & \end{tikzcd}$$

With this in place, we are ready to begin our calculation of the canonical sheaves of these threefolds. We note first that, as ϕ is a crepant morphism and f is a small resolution, we have the relations $\omega_{X^t} \cong \phi^*\omega_X$ and $\omega_Y \cong f^*\omega_{X^t}$. So we may easily derive the canonical sheaves of all of the threefolds above from the canonical sheaf of X. This is calculated by:

Theorem 5.1.1. *The canonical sheaf ω_X is given by*

$$\omega_X \cong \pi^*\big(\omega_S \otimes \det(\mathcal{E}_1) \otimes \mathcal{O}_S(\tau) \otimes (\mathcal{E}_3^+)^{-1}\big).$$

Remark 5.1.2. This formula agrees nicely with the results of Fujino and Mori [FM00] [Fuj03], who prove that the canonical divisor on X should have the form

$$K_X \sim \pi^*(K_S + D) + B,$$

for divisors D on S and B on X that satisfy certain conditions. As K_X is locally trivial in a neighbourhood of any fibre, it follows from the description of B given in [FM00] that B must vanish in our case. The theorem above then gives us that $\mathcal{O}_S(D) \cong \det(\mathcal{E}_1) \otimes \mathcal{O}_S(\tau) \otimes (\mathcal{E}_3^+)^{-1}$.

Proof. Recall from Section 4.2 that we can decompose the projection $\pi\colon X \to S$ into

$$X := \mathbf{Proj}_S\mathcal{R} \xrightarrow{\psi} \mathbf{Proj}_S\mathcal{A} \xrightarrow{\pi_{\mathcal{A}}} S,$$

where ψ is the double cover with branch locus given by Proposition 4.2.6. Define $Z := \mathbf{Proj}_S\mathcal{A}$. We will use the decomposition above to calculate the canonical sheaf of X in terms of the canonical sheaf of Z, which will turn out to be easier to calculate explicitly.

As X and Z have at worst canonical singularities, the adjunction formula for finite double covers gives

$$\omega_X \cong \psi^*\big(\omega_Z \otimes \mathcal{O}_Z(3) \otimes \pi_{\mathcal{A}}^*(\mathcal{E}_3^+)^{-1}\big). \tag{5.1}$$

Thus it just remains to calculate ω_Z. We will achieve this by embedding Z into a certain weighted projective bundle (see Section 1.2 for definitions), then using adjunction. We begin with a lemma:

Lemma 5.1.3. *Let $\mathcal{M}$ be an invertible sheaf on S. As in Definition 1.2.2, denote by $\widetilde{\mathrm{Sym}}(\mathcal{E}_1 \oplus \mathcal{M})$ the weighted symmetric algebra of $\mathcal{E}_1 \oplus \mathcal{M}$, where we insist that $\mathcal{E}_1$ (resp. $\mathcal{M}$) have homogeneous degree* 1 *(resp.* 2*) in $\widetilde{\mathrm{Sym}}(\mathcal{E}_1 \oplus \mathcal{M})$, and let $\widetilde{\mathrm{Sym}}^m(\mathcal{E}_1 \oplus \mathcal{M})$ denote its mth graded part. Then if $\mathcal{M}$ is chosen such that $\mathcal{M}^{-1}$ is sufficiently ample, there are exact sequences*

$$0 \longrightarrow \mathcal{O}_S(-\tau) \otimes \mathcal{M} \otimes \widetilde{\mathrm{Sym}}^{n-2}(\mathcal{E}_1 \oplus \mathcal{M}) \longrightarrow \widetilde{\mathrm{Sym}}^n(\mathcal{E}_1 \oplus \mathcal{M}) \longrightarrow \mathcal{A}_n \longrightarrow 0$$

for all $n \geq 2$.

Remark 5.1.4. These sequences can be used as an alternative to those given by Proposition 4.2.4 when constructing the sheaf of algebras $\mathcal{A}$. The pair $(\mathcal{E}_2, \sigma_2)$ is determined by the choice of homomorphism $\mathcal{M} \otimes \mathcal{O}_S(-\tau) \to \mathrm{Sym}^2(\mathcal{E}_1)$.

Proof. Note first that, by Lemma 4.2.3, we have that $\mathcal{A}_{2n} = \mathcal{E}_{2n}^+$ and $\mathcal{A}_{2n+1} = \mathcal{E}_{2n+1}^-$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{M}\otimes\mathcal{O}_S(-\tau) \arrow[r, equal] \arrow[d] & \mathcal{M}\otimes\mathcal{O}_S(-\tau) \arrow[d] & \\
0 \arrow[r] & \mathrm{Sym}^2(\mathcal{E}_1) \arrow[r] \arrow[d, equal] & \mathrm{Sym}^2(\mathcal{E}_1)\oplus\mathcal{M} \arrow[r] \arrow[d, "f_2"] & \mathcal{M} \arrow[r] \arrow[d, "g_2"] & 0 \\
0 \arrow[r] & \mathrm{Sym}^2(\mathcal{E}_1) \arrow[r, "\sigma_2"] & \mathcal{E}_2 \arrow[r] \arrow[d] & \mathcal{O}_\tau \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Figure 5.1.

for all $n \geq 1$. So, by the argument at the end of Section 4.1, for each $n \geq 1$ we have an exact sequence

$$0 \longrightarrow \mathrm{Sym}^n(\mathcal{E}_1) \xrightarrow{\sigma_n} \mathcal{A}_n \longrightarrow \mathcal{T}_n^{\pm} \longrightarrow 0,$$

where, by Lemma 4.2.1,

$$\mathcal{T}_n^{\pm} := \bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} \mathcal{O}_{i\tau}^{\oplus(2(n-2i)+1)}$$

We begin by proving the theorem for $n = 2$, then generalise this proof to work for higher values of n. The case $n = 2$ is proved by showing that we have a commutative diagram with exact rows and columns as shown in Figure 5.1.

Exactness of the bottom row of this diagram is given above. Furthermore, the middle row is a split exact sequence and the right-hand column is given by the tensor product of the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(-\tau) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_\tau \longrightarrow 0$$

with the invertible sheaf $\mathcal{M}$. Thus it just remains to define a morphism f_2 that makes the whole diagram commute; surjectivity of f_2 and the expression for $\ker(f_2)$ will then follow immediately from the snake lemma.

We will define f_2 separately on each of the factors of $\mathrm{Sym}^2(\mathcal{E}_1) \oplus \mathcal{M}$. It is easy to see that the restriction $f_2 \colon \mathrm{Sym}^2(\mathcal{E}_1) \to \mathcal{E}_2$ must be equal to σ_2. Then the left hand square of Figure 5.1 will obviously commute. Unfortunately, we shall see that the restriction $f_2 \colon \mathcal{M} \to \mathcal{E}_2$ is somewhat more tricky to define.

Applying the left exact functor $\mathrm{Hom}(\mathcal{M}, -)$ to the short exact sequence in the bottom row of Figure 5.1, we obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(\mathcal{M}, \mathcal{E}_2) \longrightarrow \mathrm{Hom}(\mathcal{M}, \mathcal{O}_\tau) \longrightarrow \mathrm{Ext}^1(\mathcal{M}, \mathrm{Sym}^2(\mathcal{E}_1)) \longrightarrow \cdots .$$

Furthermore, by standard properties of Ext groups [Har77, Section II.6], we have an isomorphism

$$\mathrm{Ext}^1(\mathcal{M}, \mathrm{Sym}^2(\mathcal{E}_1)) \cong H^1(S, \mathcal{M}^{-1} \otimes \mathrm{Sym}^2(\mathcal{E}_1)).$$

But if $\mathcal{M}^{-1}$ is chosen sufficiently ample, [Har77, Proposition III.5.3] shows that this cohomology group vanishes. So the morphism $\mathrm{Hom}(\mathcal{M}, \mathcal{E}_2) \to \mathrm{Hom}(\mathcal{M}, \mathcal{O}_\tau)$ is surjective. Therefore, we may find $\bar{f}_2 \in \mathrm{Hom}(\mathcal{M}, \mathcal{E}_2)$ that maps to g_2 under this morphism. Define the restriction $f_2 \colon \mathcal{M} \to \mathcal{E}_2$ to be equal to $\bar{f}_2$. Then the bottom right-hand square of Figure 5.1 commutes by construction.

Thus we see that Figure 5.1 is a commutative diagram with exact rows and columns, and the required exact sequence can be read off from the middle column.

In order to prove the result for $n > 2$, we will show that for each such n there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Sym}^n(\mathcal{E}_1) \arrow[r] \arrow[d, equal] & \widetilde{\mathrm{Sym}}^n(\mathcal{E}_1 \oplus \mathcal{M}) \arrow[r] \arrow[d, "f_n"] & \widetilde{\mathrm{Sym}}^{n-2}(\mathcal{E}_1 \oplus \mathcal{M}) \otimes \mathcal{M} \arrow[r] \arrow[d, "g_n"] & 0 \\ 0 \arrow[r] & \mathrm{Sym}^n(\mathcal{E}_1) \arrow[r, "\sigma_n"] & \mathcal{A}_n \arrow[r] & \mathcal{T}_n^{\pm} \arrow[r] & 0. \end{tikzcd} \tag{5.2}$$

Exactness of the bottom row is proved above. Thus it just remains to prove that the top row is exact and to define morphisms f_n and g_n that make the whole diagram commute.

First consider the case where n is even, $n = 2m$ ($m > 1$) say. Then there is an isomorphism

$$\widetilde{\mathrm{Sym}}^{2m}(\mathcal{E}_1 \oplus \mathcal{M}) \cong \bigoplus_{i=0}^{m} \mathrm{Sym}^{2i}(\mathcal{E}_1) \otimes \mathcal{M}^{m-i}.$$

From this expression it is immediately clear that the top row of diagram (5.2) is exact and split. We will define the maps f_{2m} and g_{2m} using the map f_2 defined above.

Note first that the restriction $f_2 \colon \mathcal{M} \to \mathcal{E}_2$ induces a morphism $\mathcal{M}^m \to \mathrm{Sym}^m(\mathcal{E}_2)$. Composing this with the map $\mathrm{Sym}^m(\mathcal{E}_2) \to \mathcal{A}_{2m}$ given by the exact sequence $(*)$ from Proposition 4.2.4, we obtain a morphism $\bar{f}_{2m} \colon \mathcal{M}^m \to \mathcal{A}_{2m}$. Define the restriction of f_{2m} to the factor $\mathrm{Sym}^{2i}(\mathcal{E}_1) \otimes \mathcal{M}^{m-i}$ for each $0 \leq i \leq m$ by

$$\begin{aligned} f_{2m,i} \colon \mathrm{Sym}^{2i}(\mathcal{E}_1) \otimes \mathcal{M}^{m-i} &\longrightarrow \mathcal{A}_{2m} \\ a \otimes b \qquad &\longmapsto \sigma_{2m}(a).\bar{f}_{2m-2i}(b), \end{aligned}$$

so that $f_{2m} := \sum_{i=0}^{m} f_{2m,i}$. Finally, define g_{2m} as the composition of the restriction $f_{2m} \colon \widetilde{\mathrm{Sym}}^{2m-2}(\mathcal{E}_1 \oplus \mathcal{M}) \otimes \mathcal{M} \to \mathcal{A}_{2m}$ with the map $\mathcal{A}_{2m} \to \mathcal{T}^{\pm}_{2m}$. Then diagram (5.2) commutes by construction.

It just remains to show that g_{2m} is surjective and to compute its kernel. In order to do this we will work locally on the stalks of the sheaves involved.

Around a point P not in the support of τ, we see that the stalk $\mathcal{T}^{\pm}_{2m,P}$ is zero, so $g_{2m,P}$ is surjective and its kernel is locally $\widetilde{\mathrm{Sym}}^{2m-2}(\mathcal{E}_{1,P} \oplus \mathcal{M}_P) \otimes \mathcal{M}_P$.

So consider a point P in the support of τ. Then, by the proof of Lemma 4.2.3, locally around P we may write

$$\mathcal{A}_P \cong \mathcal{O}_{S,P}[x_1, x_2, x_3, y]/(Q(t)),$$

where the x_i are considered to have degree 1, y has degree two and $Q(t)$ is a degree two relation between them (that depends upon a uniformising parameter t for $\mathcal{O}_{S,P}$). Note that $\mathcal{E}_{1,P}$ is generated by x_1, x_2, x_3.

Furthermore, by the proof of Lemma 4.2.1, we see that y is a local generator of $\mathcal{O}_{\tau,P}$. As we know that $g_{2,P}\colon \mathcal{M}_P \to \mathcal{O}_{\tau,P}$ is surjective, there must exist $\bar{y} \in \mathcal{M}_P$ mapping to $y \in \mathcal{O}_{\tau,P}$ under $g_{2,P}$. We will use this and the explicit description of the generators of $\mathcal{T}^{\pm}_{2m,P}$ from Lemma 4.2.1 to show that $g_{2m,P}$ is surjective for all m.

Using the direct sum decompositions of $\widetilde{\mathrm{Sym}}^{2m-2}(\mathcal{E}_{1,P} \oplus \mathcal{M})$ and $\mathcal{T}^{\pm}_{2m,P}$ given above, we see that in order to prove that $g_{2m,P}$ is surjective it is enough to show that the restrictions

$$g^i_{2m,P}\colon \mathrm{Sym}^{2i}(\mathcal{E}_1) \otimes \mathcal{M}^{m-i} \longrightarrow \mathcal{O}^{\oplus(4i+1)}_{(m-i)\tau}$$

are surjective for each $0 \leq i \leq m-1$.

By the proof of Lemma 4.2.1, we see that $\mathcal{O}^{\oplus(4i+1)}_{(m-i)\tau}$ is generated by the monomials

$$\{x_1 y^{m-i} h_{2i-1}(x_2, x_3), y^{m-i} h_{2i}(x_2, x_3)\},$$

where $h_j(x_2, x_3)$ denotes any monomial of degree j in the variables x_2, x_3. We have

$$\begin{aligned} x_1 y^{m-i} h_{2i-1}(x_2, x_3) &= g^i_{2m,P}\big(x_1 h_{2i-1}(x_2, x_3) \otimes \bar{y}^{m-i}\big), \\ y^{m-i} h_{2i}(x_2, x_3) &= g^i_{2m,P}\big(h_{2i}(x_2, x_3) \otimes \bar{y}^{m-i}\big), \end{aligned}$$

so $g^i_{2m,P}$ is surjective. Furthermore, this local description shows that the kernel of $g^i_{2m,P}$ is given by

$$\ker(g^i_{2m,P}) = \mathrm{Sym}^{2i}(\mathcal{E}_1) \otimes \mathcal{M}^{m-i} \otimes \mathcal{O}_{S,P}(-\tau).$$

Putting this together, we see that $g_{2m,P}$ is surjective and has kernel

$$\mathcal{O}_{S,P}(-\tau) \otimes \mathcal{M}_P \otimes \widetilde{\mathrm{Sym}}^{2m-2}(\mathcal{E}_{1,P} \oplus \mathcal{M}_P).$$

This proves that g_{2m} is surjective and has kernel $\mathcal{O}_S(-\tau) \otimes \mathcal{M} \otimes \widetilde{\mathrm{Sym}}^{2m-2}(\mathcal{E}_1 \oplus \mathcal{M})$, so by the snake lemma we see that f_{2m} is also surjective with the same kernel. This proves the lemma in the case where n is even.

The case where $n = 2m+1$ ($m \geq 1$) is odd is proved in a very similar way. We use the decomposition

$$\widetilde{\mathrm{Sym}}^{2m+1}(\mathcal{E}_1 \oplus \mathcal{M}) \cong \bigoplus_{i=0}^{m} \mathrm{Sym}^{2i+1}(\mathcal{E}_1) \otimes \mathcal{M}^{m-i}$$

and the morphisms $\sigma_{2i+1} \colon \mathrm{Sym}^{2i+1}(\mathcal{E}_1) \to \mathcal{A}_{2i+1}$ and $\bar{f}_{2m-2i} \colon \mathcal{M}^{m-i} \to \mathcal{A}_{2m-2i}$ to define f_{2m+1} and g_{2m+1} as before. Surjectivity of these maps and the expressions for their kernels are then proved exactly as above.

This completes the proof of Lemma 5.1.3. □

From this lemma we obtain an embedding of Z into the weighted projective bundle $\tilde{\mathbb{P}} := \tilde{\mathbb{P}}_S(\mathcal{E}_1 \oplus \mathcal{M})$, where $\mathcal{E}_1 \oplus \mathcal{M}$ is considered as a weighted locally free sheaf with weights $(1,2)$ on S (for definitions see Section 1.2). Let $\pi_{\tilde{\mathbb{P}}} \colon \tilde{\mathbb{P}} \to S$ denote the natural projection. Then we see that Z is a divisor in the linear system $|\mathcal{O}_{\tilde{\mathbb{P}}}(2) \otimes \pi_{\tilde{\mathbb{P}}}^*(\mathcal{O}_S(\tau) \otimes \mathcal{M}^{-1})|$ on $\tilde{\mathbb{P}}$. So the adjunction formula gives

$$\omega_Z \cong \omega_{\tilde{\mathbb{P}}}|_Z \otimes \pi_{\mathcal{A}}^*\big(\mathcal{O}_S(\tau) \otimes \mathcal{M}^{-1}\big) \otimes \mathcal{O}_Z(2). \tag{5.3}$$

All that remains is to find an expression for $\omega_{\tilde{\mathbb{P}}}$. In order to do this we use the following exact sequence, obtained as a relative version of [Dol82, 2.1.5]

$$0 \longrightarrow \Omega^1_{\tilde{\mathbb{P}}/S} \longrightarrow \pi_{\tilde{\mathbb{P}}}^*\big(\mathcal{E}_1(-1) \oplus \mathcal{M}(-2)\big) \longrightarrow \mathcal{O}_{\tilde{\mathbb{P}}} \longrightarrow 0.$$

Taking the top wedge power of the sheaves in this sequence gives an expression

$$\omega_{\tilde{\mathbb{P}}/S} \cong \pi_{\tilde{\mathbb{P}}}^*\big(\det(\mathcal{E}_1) \otimes \mathcal{M}\big) \otimes \mathcal{O}_{\tilde{\mathbb{P}}}(-5).$$

Finally, noting that $\omega_{\tilde{\mathbb{P}}/S} \cong \omega_{\tilde{\mathbb{P}}} \otimes \pi_{\tilde{\mathbb{P}}}^* \omega_S^{-1}$, back-substitution into equations (5.3) and (5.1) gives the result. $\square$

In light of this theorem, we define an invertible sheaf on S

$$\mathcal{K} := \omega_S \otimes \det(\mathcal{E}_1) \otimes \mathcal{O}_S(\tau) \otimes (\mathcal{E}_3^+)^{-1},$$

so that $\omega_X \cong \pi^*\mathcal{K}$. The sheaf $\mathcal{K}$ will turn out to be important in many of the calculations that follow in the remainder of this chapter.

We conclude this section with the following result that calculates the Kodaira dimension of X. As the Kodaira dimension is a birational invariant, this will coincide with the Kodaira dimensions of X^t and Y.

Corollary 5.1.5. *The Kodaira dimension $\kappa(X)$ is given by*

$$\kappa(X) = \begin{cases} 1 & \text{if } \deg(\mathcal{K}) > 0, \\ 0 & \text{if } \mathcal{K}^n \cong \mathcal{O}_S \text{ for some } n > 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. By definition, the Kodaira dimension is given as the smallest integer $\kappa(X)$ such that

$$h^0(X, \omega_X^n) \leq (\text{const.}) n^{\kappa(X)}$$

for all $n \geq 1$, where we adopt the convention that if $h^0(X, \omega_X^n) = 0$ for all $n \geq 1$ then $\kappa(X) := -\infty$.

Now Theorem 5.1.1 gives us that $\omega_X \cong \pi^*\mathcal{K}$, so by the projection formula and the Leray spectral sequence

$$h^0(X, \omega_X^n) \cong h^0(S, \mathcal{K}^n).$$

The result then follows immediately from [Har77, Remark IV.1.3.2]. $\square$

5.2 The Coherent Euler Characteristic

The aim of this section is to calculate the coherent Euler characteristics $\chi(\mathcal{O}_X)$, $\chi(\mathcal{O}_{X^t})$ and $\chi(\mathcal{O}_Y)$ for the threefolds X, X^t and Y defined in the last section. We have:

Theorem 5.2.1. *The coherent Euler characteristics $\chi(\mathcal{O}_X)$, $\chi(\mathcal{O}_{X^t})$ and $\chi(\mathcal{O}_Y)$ are all equal and given by $-\deg(\mathcal{K})$, where $\mathcal{K}$ is the invertible sheaf on S defined in the last section.*

Proof. We perform the calculation for X, the calculations for X^t and Y are identical.

So let X be as usual and let $\mathcal{L} := \mathcal{O}_X(1) \otimes \omega_X^{-1}$ be the polarisation sheaf on X. Note that, by the proof of Theorem 4.3.2, the sheaf $\mathcal{L}$ is invertible. In order to calculate $\chi(\mathcal{O}_X)$, we use the Riemann-Roch theorem for singular threefolds [Rei87, Theorem 10.2]. This states that, for D a Weil divisor on X, there is an expression

$$\chi\big(X, \mathcal{O}_X(D)\big) = \chi(\mathcal{O}_X) + \frac{1}{12}D.(D - K_X).(2D - K_X) + \frac{1}{12}D.c_2(X) + \sum_Q c_Q(D),$$

where $c_2(X)$ denotes the second Chern class of X and the summation takes place over the singularities Q of the sheaf $\mathcal{O}_X(D)$, where $c_Q(D)$ denotes a contribution due to the singularity at Q.

We will apply this theorem to the sheaves $\mathcal{O}_X(D) := \omega_X^n \otimes \mathcal{L}^n$ for $n \in \{1, 2, 3\}$. Note first that, as ω_X and $\mathcal{L}$ are both invertible sheaves on X, these sheaves will be nonsingular and there will be no contributions $c_Q(D)$. Next, by Theorem 5.1.1, ω_X is the inverse image of the sheaf $\mathcal{K}$ on S, so the intersection numbers $\omega_X.\omega_X.\omega_X = 0$ and $\omega_X.\omega_X.\mathcal{L} = 0$. Furthermore, as the restriction of $\mathcal{L}$ to a fibre has self-intersection number 2, we must have $\omega_X.\mathcal{L}.\mathcal{L} = 2\deg(\mathcal{K})$.

Substituting in to the equation above, we obtain

$$\chi(X, \omega_X \otimes \mathcal{L}) = \chi(\mathcal{O}_X) + \frac{1}{2}\deg(\mathcal{K}) + \frac{1}{6}\,\mathcal{L}.\mathcal{L}.\mathcal{L} + \frac{1}{12}\,(\omega_X \otimes \mathcal{L}).c_2(X)$$

$$\chi(X, \omega_X^2 \otimes \mathcal{L}^2) = \chi(\mathcal{O}_X) + 6\deg(\mathcal{K}) + \frac{4}{3}\,\mathcal{L}.\mathcal{L}.\mathcal{L} + \frac{1}{6}\,(\omega_X \otimes \mathcal{L}).c_2(X)$$
$$\chi(X, \omega_X^3 \otimes \mathcal{L}^3) = \chi(\mathcal{O}_X) + \frac{45}{2}\deg(\mathcal{K}) + \frac{9}{2}\,\mathcal{L}.\mathcal{L}.\mathcal{L} + \frac{1}{4}\,(\omega_X \otimes \mathcal{L}).c_2(X).$$

In order to prove the theorem we will derive expressions for $\chi(X, \omega_X^n \otimes \mathcal{L}^n)$ for each $n \in \{1, 2, 3\}$, then solve the above equations simultaneously to find $\chi(\mathcal{O}_X)$.

Observe first that as $\mathcal{L}$ is π-ample on X and ω_X is the inverse image of a sheaf under π, by [KMM87, Theorem 1.2.5] we have that the higher direct images $R^i\pi_*(\omega_X^n \otimes \mathcal{L}^n) = 0$ for all $i > 0$ and all $n > 0$ (we note here that different vanishing results must be used here when calculating $\chi(\mathcal{O}_{X^t})$ and $\chi(\mathcal{O}_Y)$: for X^t use the extension of the above result to π-nef and π-big divisors [KMM87, Remark 1.2.6] and for Y use the analytic relative vanishing theorem of Ancona [Anc87, Theorem 2.1]).

Using this, by the Leray spectral sequence we see that

$$\begin{aligned}\chi\big(X, \omega_X^n \otimes \mathcal{L}^n\big) &= \chi\big(S, \pi_*(\omega_X^n \otimes \mathcal{L}^n)\big)\\ &= \chi\big(S, \pi_*\mathcal{O}_X(1)^n\big)\\ &= \chi(S, \mathcal{E}_n)\end{aligned}$$

for all $n > 0$.

Now, by definition, $\chi(S, \mathcal{E}_1) = \deg(\mathcal{E}_1) + 3\chi(\mathcal{O}_S)$. Furthermore, the exact sequence

$$0 \longrightarrow \mathrm{Sym}^2(\mathcal{E}_1) \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{O}_\tau \longrightarrow 0$$

gives $\deg(\mathcal{E}_2) = \deg(\tau) + \deg(\mathrm{Sym}^2(\mathcal{E}_1))$. Then, using the exact sequence

$$0 \longrightarrow \bigwedge^2 \mathcal{E}_1 \longrightarrow \mathcal{E}_1 \otimes \mathcal{E}_1 \longrightarrow \mathrm{Sym}^2(\mathcal{E}_1) \longrightarrow 0$$

and the isomorphism $\bigwedge^2 \mathcal{E}_1 \cong \det(\mathcal{E}_1) \otimes \mathcal{E}_1^\vee$, we obtain

$$\begin{aligned}
\deg\big(\mathrm{Sym}^2(\mathcal{E}_1)\big) &= \deg(\mathcal{E}_1 \otimes \mathcal{E}_1) - \deg\Big(\bigwedge^2 \mathcal{E}_1\Big) \\
&= 6\deg(\mathcal{E}_1) - \deg\big(\det(\mathcal{E}_1) \otimes \mathcal{E}_1^\vee\big) \\
&= 4\deg(\mathcal{E}_1).
\end{aligned}$$

So $\deg(\mathcal{E}_2) = 4\deg(\mathcal{E}_1) + \deg(\tau)$, giving

$$\chi(X, \mathcal{E}_2) = 4\deg(\mathcal{E}_1) + \deg(\tau) + 6\chi(\mathcal{O}_s).$$

It just remains to find $\deg(\mathcal{E}_3)$. Lemma 4.2.1 gives an exact sequence

$$0 \longrightarrow \mathrm{Sym}^3(\mathcal{E}_1) \longrightarrow \mathcal{E}_3 \longrightarrow \mathcal{E}_3^+ \oplus \mathcal{O}_\tau^{\oplus 3} \longrightarrow 0$$

which in turn gives

$$\deg(\mathcal{E}_3) = \deg\big(\mathrm{Sym}^3(\mathcal{E}_1)\big) + \deg(\mathcal{E}_3^+) + 3\deg(\tau).$$

To find $\deg(\mathrm{Sym}^3(\mathcal{E}_1))$ we use the exact sequence

$$0 \longrightarrow \det(\mathcal{E}_1) \longrightarrow \big(\mathcal{E}_1 \otimes (\mathcal{E}_1 \wedge \mathcal{E}_1)\big)^{\oplus 2} \longrightarrow \mathcal{E}_1^{\otimes 3} \longrightarrow \mathrm{Sym}^3(\mathcal{E}_1) \longrightarrow 0,$$

which gives

$$\begin{aligned}
\deg\big(\mathrm{Sym}^3(\mathcal{E}_1)\big) &= \deg(\mathcal{E}_1^{\otimes 3}) + \deg\big(\det(\mathcal{E}_1)\big) - \deg\big((\mathcal{E}_1 \otimes (\mathcal{E}_1 \wedge \mathcal{E}_1))^{\oplus 2}\big) \\
&= 27\deg(\mathcal{E}_1) + \deg(\mathcal{E}_1) - 6\deg(\mathcal{E}_1 \wedge \mathcal{E}_1) - 6\deg(\mathcal{E}_1) \\
&= 10\deg(\mathcal{E}_1).
\end{aligned}$$

Thus

$$\deg(\mathcal{E}_3) = 10\deg(\mathcal{E}_1) + \deg(\mathcal{E}_3^+) + 3\deg(\tau),$$

so

$$\chi(X, \mathcal{E}_3) = 10\deg(\mathcal{E}_1) + \deg(\mathcal{E}_3^+) + 3\deg(\tau) + 11\chi(\mathcal{O}_S).$$

Substituting these expressions into the Riemann-Roch formulae above and solving simultaneously we obtain

$$\begin{aligned}(\omega_X \otimes \mathcal{L}).c_2(X) &= 4\deg(\mathcal{E}_3^+) + 26\deg(\mathcal{E}_1) + 16\deg(\tau)\\ \mathcal{L}.\mathcal{L}.\mathcal{L} &= 6\deg(\mathcal{E}_3^+) - 4\deg(\mathcal{E}_1) - 5\deg(\tau) - 6\deg(\omega_S)\\ \chi(\mathcal{O}_X) &= \deg(\mathcal{E}_3^+) - \deg(\mathcal{E}_1) - \deg(\tau) - \deg(\omega_S) = -\deg(\mathcal{K}).\end{aligned}$$

This completes the proof of Theorem 5.2.1. □

5.3 Constructing Calabi-Yau Threefolds

In most of the rest of this chapter we will be concerned with the case when the analytic threefold Y (defined in Section 5.1) is a *Calabi-Yau threefold*. Calabi-Yau manifolds are a higher dimensional generalisation of K3 surfaces that are of particular interest to researchers and K3-fibrations can provide a good way to construct them. In particular, we note that K3-fibred Calabi-Yau manifolds have been studied in relation to some versions of string theory (for instance, see [KLM95]). For this reason it would be good to know whether our construction can produce them and what properties any that are constructed can have.

Definition 5.3.1. *A* Calabi-Yau threefold *is a nonsingular three dimensional compact Kähler manifold Y, with vanishing canonical bundle $\omega_Y \cong \mathcal{O}_Y$ and trivial cohomology groups $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$.*

With this in place, we can give conditions under which our construction produces a Calabi-Yau threefold:

Theorem 5.3.2. *Y is a Calabi-Yau threefold if and only if it is Kähler and*

- *the base curve $S \cong \mathbb{P}^1$,*
- *the invertible sheaf $\mathcal{K} \cong \mathcal{O}_S$.*

Proof. First suppose that Y is a Calabi-Yau threefold. Then $\omega_Y \cong \mathcal{O}_Y \cong \pi_Y^*\mathcal{K}$ by Theorem 5.1.1, so the projection formula gives $\mathcal{K} \cong \mathcal{O}_S$. Furthermore, the Leray spectral sequence gives an injective homomorphism $H^1(S, \mathcal{O}_S) \to H^1(Y, \mathcal{O}_Y)$, which implies that $h^1(Y, \mathcal{O}_Y) \geq g(S)$. Thus $g(S) = 0$ and $S \cong \mathbb{P}^1$.

Next suppose that we have constructed $\pi_Y \colon Y \to S \cong \mathbb{P}^1$ satisfying the conditions of the theorem. Then Theorem 5.1.1 gives $\omega_Y \cong \pi_Y^*\mathcal{K} \cong \mathcal{O}_Y$. Furthermore, Serre duality gives $H^1(Y, \mathcal{O}_Y) \cong H^2(Y, \mathcal{O}_Y)$. So in order to prove the theorem it suffices to prove that $H^1(Y, \mathcal{O}_Y) = 0$.

The Leray spectral sequence gives rise to an exact sequence

$$0 \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow H^0(S, R^1\pi_{Y*}\mathcal{O}_Y) \longrightarrow H^2(S, \mathcal{O}_S) \longrightarrow \cdots .$$

Using the fact that $S \cong \mathbb{P}^1$, we obtain an isomorphism

$$H^1(Y, \mathcal{O}_Y) \cong H^0(S, R^1\pi_{Y*}\mathcal{O}_Y).$$

Thus, it suffices to show that $R^1\pi_{Y*}\mathcal{O}_Y = 0$.

In order to prove this, we will show that $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$ for any fibre Y_0 of π_Y. The vanishing of $R^1\pi_{Y*}\mathcal{O}_Y$ will then follow from a corollary of the theorem on cohomology and base change [BS76, Corollary 3.5].

Lemma 5.3.3. *$H^1(Y_0, \mathcal{O}_{Y_0}) = 0$ for any fibre Y_0 of $\pi_Y \colon Y \to S$.*

Proof. Let Y_0 be any fibre of $\pi_Y : Y \to S$. Then as $\pi_Y : Y \to S$ is a semistable analytic threefold fibred by K3 surfaces, Y_0 is one of the fibres from the classification of Theorem 2.3.5. We will analyse each of the cases in this classification in turn.

If Y_0 is a fibre of Type I then it is a smooth K3 surface, so $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$ by definition. Next suppose that Y_0 is a fibre of Type II. Recall then that Y_0 is a chain of surfaces $V_0 \cup \cdots \cup V_r$, for rational surfaces V_0, V_r and elliptic ruled surfaces $V_1, \ldots, V_{r-1}$, with $V_{i-1} \cap V_i = D_i$ smooth elliptic. Furthermore, by Theorem 3.4.1 we may assume that $V_1, \ldots, V_{r-1}$ are minimal.

Using an argument based upon that used to prove [SB83b, Lemma 2.12], we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{V_0}(-D_1) \longrightarrow \mathcal{O}_{Y_0} \longrightarrow \mathcal{O}_{Y_0 - V_0} \longrightarrow 0.$$

By Lemma 3.3.6 we have $K_{V_0} \sim -D_1$ on V_0, so by Serre duality and the properties of rational surfaces we have $H^1(V_0, \mathcal{O}_{V_0}(-D_1)) = 0$. Thus, the long exact sequence of cohomology associated to the above short exact sequence gives an injection

$$H^1(Y_0, \mathcal{O}_{Y_0}) \hookrightarrow H^1(Y_0 - V_0, \mathcal{O}_{Y_0 - V_0}).$$

Now we proceed by induction on the components of Y_0. For $0 \leq k < r$, write $\Sigma_k := V_0 \cup \cdots \cup V_k$. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{V_k}(-D_{k+1}) \longrightarrow \mathcal{O}_{Y_0 - \Sigma_{k-1}} \longrightarrow \mathcal{O}_{Y_0 - \Sigma_k} \longrightarrow 0$$

for all $0 < k < r$. We wish to prove that the long exact sequence of cohomology associated to this short exact sequence gives rise to an injection

$$H^1(Y_0 - \Sigma_{k-1}, \mathcal{O}_{Y_0 - \Sigma_{k-1}}) \hookrightarrow H^1(Y_0 - \Sigma_k, \mathcal{O}_{Y_0 - \Sigma_k})$$

To show this, it suffices to prove that $H^1(V_k, \mathcal{O}_{V_k}(-D_{k+1})) = 0$.

So suppose first that $(D_{k+1}|_{V_k})^2 > 0$. Then $D_{k+1}|_{V_k}$ is nef and big, so by the general Kodaira vanishing theorem [KM98, Theorem 2.70] we have $H^1(V_k, \mathcal{O}_{V_k}(-D_{k+1})) = 0$. Furthermore, if $(D_{k+1}|_{V_k})^2 < 0$ then, by Lemma 3.3.10, we must have $(D_k|_{V_k})^2 > 0$ and $K_{V_k} \sim -D_k|_{V_k} - D_{k+1}|_{V_k}$, so by Serre duality and the general Kodaira vanishing theorem,

$$H^1(V_k, \mathcal{O}_{V_k}(-D_{k+1})) = H^1(V_k, \mathcal{O}_{V_k}(-D_k)) = 0.$$

Finally, in the case where $(D_{k+1}|_{V_k})^2 = 0$, let F denote any fibre of the ruling and consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{V_k}(-D_{k+1} - F) \longrightarrow \mathcal{O}_{V_k}(-D_{k+1}) \longrightarrow \mathcal{O}_F(-D_{k+1}) \longrightarrow 0.$$

By the explicit description of $\mathrm{Pic}(V_i)$ in Lemma 3.3.9, the divisor $(D_{k+1} + F)$ is nef and big, so by the general Kodaira vanishing theorem we have $H^1(V_k, \mathcal{O}_{V_k}(-D_{k+1} - F)) = 0$. Moreover $F \cong \mathbb{P}^1$ and $\mathcal{O}_F(-D_{k+1}) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, so $H^1(F, \mathcal{O}_F(-D_{k+1})) = 0$. So, by the long exact sequence of cohomology associated to this short exact sequence, we see that $H^1(V_k, \mathcal{O}_{V_k}(-D_{k+1})) = 0$, as required.

Thus we have injections $H^1(Y_0 - \Sigma_{k-1}, \mathcal{O}_{Y_0 - \Sigma_{k-1}}) \hookrightarrow H^1(Y_0 - \Sigma_k, \mathcal{O}_{Y_0 - \Sigma_k})$ for all $0 < k < r$. These compose to give an injection

$$H^1(Y_0, \mathcal{O}_{Y_0}) \hookrightarrow H^1(V_r, \mathcal{O}_{V_r}).$$

But V_r is rational, so $H^1(V_r, \mathcal{O}_{V_r}) = 0$. Therefore, $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$ as well.

Finally, we consider the case when Y_0 is a fibre of Type III. The proof that the first cohomology $H^1(Y_0, \mathcal{O}_{Y_0})$ vanishes in this case will be quite similar to that used in the Type II case. Recall that Y_0 consists of rational surfaces meeting along rational curves that form cycles in each component, such that the dual graph Γ of Y_0 is a triangulation of the sphere S^2.

Begin by ordering the vertices $V_0, \ldots, V_r$ of Γ such that the subgraphs spanned by the sets of vertices $\{V_0, \ldots, V_k\}$ and $\{V_{k+1}, \ldots, V_r\}$ are connected for all $0 \leq k < r$. Let Σ_k denote the set of components of Y_0 corresponding to the set of vertices $\{V_0, \ldots, V_k\}$; these components will also be labelled $V_0, \ldots V_k$. Then $\Sigma_k - \Sigma_{k-1} = V_k$ and, since S^2 is simply connected, $V_k \cap \Sigma_{k-1}$ is a connected subset of the cycle of double curves on V_k.

By the same argument as before, we see that we have an injective homomorphism

$$H^1(Y_0, \mathcal{O}_{Y_0}) \hookrightarrow H^1(Y_0 - \Sigma_0, \mathcal{O}_{Y_0 - \Sigma_0}).$$

We would like to show that, for all $1 < k < r$, there are injective homomorphisms

$$H^1(Y_0 - \Sigma_{k-1}, \mathcal{O}_{Y_0 - \Sigma_{k-1}}) \hookrightarrow H^1(Y_0 - \Sigma_k, \mathcal{O}_{Y_0 - \Sigma_k}).$$

Composing these gives an injective homomorphism $H^1(Y_0, \mathcal{O}_{Y_0}) \hookrightarrow H^1(V_r, \mathcal{O}_{V_r})$, from which $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$ will follow immediately from the fact that V_r is rational.

To show that we have the injective homomorphisms above, we once again consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{V_k}(-D_k) \longrightarrow \mathcal{O}_{Y_0 - \Sigma_{k-1}} \longrightarrow \mathcal{O}_{Y_0 - \Sigma_k} \longrightarrow 0,$$

where $D_k = V_k \cap (Y_0 - \Sigma_k)$ is a connected chain (or cycle) of rational curves on V_k. We wish to show that $H^1(V_k, \mathcal{O}_{V_k}(-D_k)) = 0$.

Note that if D_k contains all double curves on V_k then by Lemma 3.3.6 we must have $D_k \sim -K_{V_i}$, so by Serre duality

$$H^1(V_k, \mathcal{O}_{V_k}(-D_k)) = H^1(V_k, \mathcal{O}_{V_k}) = 0.$$

Thus we may assume that D_k does not contain all double curves on V_k, so is a connected chain of rational curves.

Now consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{V_k}(-D_k) \longrightarrow \mathcal{O}_{V_k} \longrightarrow \mathcal{O}_{D_k} \longrightarrow 0.$$

As V_k is rational, the cohomology groups $H^1(V_k, \mathcal{O}_{V_k})$ and $H^2(V_k, \mathcal{O}_{V_k})$ both vanish. Furthermore, as D_k is a connected chain of rational curves, we have $h^0(D_k, \mathcal{O}_{D_k}) = 1$ and $H^1(D_k, \mathcal{O}_{D_k}) = 0$. So the long exact sequence of cohomology associated to the above short exact sequence gives $H^2(V_k, \mathcal{O}_{V_k}(-D_k)) = 0$. Now, by Serre duality we have

$$H^0(V_k, \mathcal{O}_{V_k}(K_{V_k} + D_k)) = H^2(V_k, \mathcal{O}_{V_k}(-D_k)) = 0.$$

But, by Lemma 3.3.6, $K_{V_k} + D_k$ is linearly equivalent to $-D_k'$, where D_k' denotes the sum of the double curves not in D_k (which is also a connected chain of rational curves). So, by symmetry of this argument in D_k and D_k', we see that $H^0(V_k, \mathcal{O}_{V_k}(-D_k)) = 0$. Putting all of this into the long exact sequence of cohomology associated to the short exact sequence above we obtain an exact sequence

$$0 \longrightarrow H^0(V_k, \mathcal{O}_{V_k}) \longrightarrow H^0(D_k, \mathcal{O}_{D_k}) \longrightarrow H^1(V_k, \mathcal{O}_{V_k}(-D_k)) \longrightarrow 0.$$

But $H^0(V_k, \mathcal{O}_{V_k})$ and $H^0(D_k, \mathcal{O}_{D_k})$ both have dimension 1, as V_k and D_k are connected, so we must have $H^1(V_k, \mathcal{O}_{V_k}(-D_k)) = 0$.

Thus we have the required injective homomorphisms, which compose to give an injective homomorphism $H^1(Y_0, \mathcal{O}_{Y_0}) \to H^1(V_r, \mathcal{O}_{V_r})$. Therefore, as V_r as rational, we immediately obtain that $H^1(Y, \mathcal{O}_Y) = 0$, as required. This completes the proof of Lemma 5.3.3. $\square$

Using this lemma, [BS76, Corollary 3.5] shows that $R^1\pi_{Y*}\mathcal{O}_Y = 0$, so by the Leray spectral sequence we obtain $H^1(Y, \mathcal{O}_Y) = 0$. This completes the proof of Theorem 5.3.2. $\square$

Two interesting numbers associated to a Calabi-Yau threefold Y are the ranks of the second and third integral cohomology groups $h^2(Y, \mathbb{Z})$ and $h^3(Y, \mathbb{Z})$. From these two values all of the Hodge numbers can be calculated; these are of particular interest to researchers studying mirror symmetry and the moduli of Calabi-Yau threefolds.

One could hope to calculate these numbers by relating them to some properties of the relative log canonical model X, which can in turn be calculated using the explicit description of X given in Chapter 4; indeed, this is what we will do for $h^2(Y, \mathbb{Z})$ in the last two sections of this chapter.

Unfortunately, the rank of the third cohomology group is much more difficult to calculate. Caibăr proves a result [Cai99, Theorem 4.22] that performs this calculation locally above an isolated singularity of X, but this result is quite tricky to use in practice (and the singularities of X may not all be isolated!). He conjectures a stronger result [Cai99, Conjecture 4.23] which, if found to be true, could be useful in calculating $h^3(Y, \mathbb{Z})$; however, to our knowledge this conjecture has yet to be proved.

5.4 The Number of Crepant Divisors

In order to perform the calculation of $h^2(Y, \mathbb{Z})$ alluded to at the end of the last section, we make use of the close relationship between the cohomology group $H^2(Y, \mathbb{Z})$ and the Picard group $\mathrm{Pic}(Y) \cong H^1(Y, \mathcal{O}_Y^*)$, given by the exponential exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y^* \longrightarrow 0.$$

This sequence gives rise to the long exact sequence of cohomology

$$\cdots \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow \mathrm{Pic}(Y) \longrightarrow H^2(Y, \mathbb{Z}) \longrightarrow H^2(Y, \mathcal{O}_Y) \longrightarrow \cdots.$$

If Y is a Calabi-Yau threefold, by definition the groups $H^1(Y, \mathcal{O}_Y)$ and $H^2(Y, \mathcal{O}_Y)$ both vanish, so $H^2(Y, \mathbb{Z}) \cong \mathrm{Pic}(Y)$. Thus, we can use the structure of the Picard group $\mathrm{Pic}(Y)$ to calculate $h^2(Y, \mathbb{Z})$.

In order to gain access the structure of $\mathrm{Pic}(Y)$, we would like to compare it to the divisor class group of the relative log canonical model X, which we should be able to calculate using the explicit description of X in Chapter 4. To make this comparison we will need to know how many divisors are contracted by the morphism $\phi_Y \colon Y \to X$.

Recall that we can factor this morphism through the terminal threefold fibred by K3 surfaces of degree two X^t. We have $\phi_Y = \phi \circ f$, where

$$Y \xrightarrow{f} X^t \xrightarrow{\phi} X.$$

As f is a small analytic resolution of the singularities of X^t, the number of divisors contracted by ϕ_Y will be the same as the number of divisors contracted by ϕ.

Now for any partial resolution $\varphi \colon Z \to X$, where Z has at most terminal singularities, recall that we may write

$$K_Z \equiv \varphi^* K_X + \sum_i a_i E_i,$$

where the sum runs over all exceptional divisors $E_i \subset Z$. As X has canonical singularities, the discrepancies $a_i \geq 0$. If $a_i = 0$ for some i, the corresponding exceptional divisor E_i is called *crepant*. The number of crepant divisors

$$c(X) := \#\{i : a_i = 0\}$$

is finite [Rei80, Lemma 2.3] and is independent of the choice of Z.

As X^t has terminal singularities, we may choose $Z = X^t$ in this definition. Furthermore, as $\phi \colon X^t \to X$ is a crepant morphism, all of the ϕ-exceptional divisors in X^t are crepant divisors. So the number of divisors contracted by ϕ is exactly $c(X)$.

It turns out that $c(X)$ is not just useful for comparing divisor groups; it is also an important invariant of X, measuring how far X is from being terminal. Unfortunately, it seems that any closed expression for $c(X)$ in terms of the data used to construct X and the classification of singular fibres within it (given by Theorem 3.2.2) is likely to be too complex to be practically computed. Instead, in this section we will give an algorithmic method by which $c(X)$ may be explicitly calculated for a given X.

We begin with a brief discussion of the singularities that can occur in X. By construction, these are all Gorenstein and canonical. Let $x \in X$ be any such singular point. Then, by [Rei80, Corollary 2.10], a general hyperplane section $x \in H \subset X$ has either a rational double point, simple elliptic or cusp singularity (see Definition 3.1.7). Recall from Definition 4.3.6 that if $x \in H$ is a rational double point, $x \in X$ is called *compound Du Val*. By [KM98, Corollary 5.40], all but finitely many of the singularities occurring in X are compound Du Val.

We are now ready to begin calculating the contribution to $c(X)$ from a singularity $x \in X$. Suppose first that $x \in X$ is not isolated. Then x lies on an irreducible curve C of singularities in X, the general point of which is compound Du Val. As a general fibre of $\pi\colon X \to S$ is smooth, C must further lie within a fibre of π. Therefore, C must coincide with a component of a double curve in a fibre of Type II or III (in the classification of Theorem 3.2.2). As the analytic resolution $\phi_Y\colon Y \to X$ contracts a chain of r_C rational or elliptic ruled components to C, this curve will have singularities of cA_{r_C} type (i.e. the general hyperplane section has an A_{r_C} singularity). So C gives rise to a chain of r_C crepant divisors in the partial resolution $\phi\colon X^t \to X$, and thus contributes r_C to $c(X)$.

After resolving the singularities along such curves, we are left with a threefold X' having only isolated singularities. Let $x \in X'$ be any such singularity and consider the case where $x \in X'$ is not compound Du Val. This situation can only occur away from the double curve in a degenerate fibre of Type (II.0), (II.1b), (II.4b) or (III.0), or on the double curve in a fibre of Type III (in the classification of Theorem 3.2.2). Furthermore, by the local description of these singularities in Section 3.1, all can be written locally as hypersurfaces in $\mathbb{C}^4$. So we may calculate their contributions to $c(X)$ by the toric methods of Caibăr [Cai99, Section 3].

After all such singularities have been resolved, any remaining singularities must be isolated and compound Du Val. But then, by [KM98, Corollary 5.38], the resulting threefold is terminal, so does not require any further resolution.

Thus, in summary, we have the following algorithm for calculating $c(X)$:

(1) If X has any non-isolated singularities then they must lie along irreducible curves C_i of singularities of cA_{r_i} type, lying along double curves in degenerate fibres of Type II or III. Each such curve contributes r_i to $c(X)$.

(2) Once all non-isolated singularities have been resolved, use the toric methods of Caibăr [Cai99, Section 3] to calculate the contributions to $c(X)$ made by each of the non-compound Du Val points.

5.5 The Second Betti Number of a Calabi-Yau Threefold

In this section we will complete the calculation of the rank of the second integral cohomology group $h^2(Y, \mathbb{Z})$ when Y is a Calabi-Yau threefold.

So let X and Y be defined as in Section 5.1, and assume that Y satisfies the conditions in Theorem 5.3.2. Then Y is a Calabi-Yau threefold fibred by K3 surfaces of degree two and X is the relative log canonical model of Y. As usual, ϕ_Y will denote the birational morphism from Y to X.

We are interested in calculating the rank of the second integral cohomology group $h^2(Y, \mathbb{Z})$. In order to do this we will use the fact that $H^2(Y, \mathbb{Z}) \cong \mathrm{Pic}(Y)$, and calculate the rank of this group by comparing it to the (Weil) divisor class group $\mathrm{Cl}(X)$. We have:

Proposition 5.5.1. *With X and Y as above, we have*

$$h^2(Y, \mathbb{Z}) = \rho(X) + c(X)$$

where $\rho(X)$ denotes the rank of $\mathrm{Cl}(X)$ and $c(X)$ denotes the number of crepant divisors calculated in Section 5.4.

Proof. By [Cai99, Proposition 4.11], there is a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & \mathrm{WDiv}(Y) \arrow[r, "\phi_{Y*}"] \arrow[d] & \mathrm{WDiv}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & \mathrm{Cl}(Y) \arrow[r, "\phi_{Y*}"] & \mathrm{Cl}(X) \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{WDiv}(Y)$ denotes the group of Weil divisors on Y and K is the subgroup of $\mathrm{WDiv}(Y)$ with support contained in $\mathrm{Ex}(\phi_Y)$. Furthermore, K is finitely generated and free.

By this description we must have $K \cong \bigoplus_i \mathbb{Z}[E_i]$, where the sum runs over the set of exceptional divisors E_i of ϕ_Y. As there are precisely $c(X)$ such divisors, we see that $\mathrm{rank}(\mathrm{Cl}(Y)) = c(X) + \rho(X)$. But, as Y is a smooth Calabi-Yau threefold, $\mathrm{Cl}(Y) \cong \mathrm{Pic}(Y) \cong H^2(Y, \mathbb{Z})$. □

Finally, it just remains to calculate $\rho(X)$. This will follow from the explicit description of X in Chapter 4. We have:

Theorem 5.5.2. *Let X be defined as above. Then*

$$\rho(X) = r(X) + 2,$$

where $r(X)$ denotes the number of reducible fibres (i.e. fibres of Types (II.3), (II.4), (III.3) *and* (III.4) *in the classification of Theorem 3.2.2) appearing in $\pi \colon X \to S$.*

Proof. Let D be any Weil divisor on X and let F be a general fibre of $\pi\colon X \to S$. Then X is nonsingular in a neighbourhood of F, so the restricted sheaf $\mathcal{O}_F(D)$ is invertible on F. As F can be seen as a sextic hypersurface in $\mathbb{P}(1,1,1,3)$, its Picard group $\mathrm{Pic}(F)$ has rank 1 and is generated by the invertible sheaf $\mathcal{O}_F(1)$. So $\mathcal{O}_F(D) \cong \mathcal{O}_F(n)$, for some integer n.

Next, as $\mathcal{O}_{\mathbb{P}^1}(1)$ is ample, we may find an integer $m > 0$ such that the sheaf $\pi_*(\mathcal{O}_X(D) \otimes \mathcal{O}_X(-n)) \otimes \mathcal{O}_S(m)$ is generated by its global sections. Furthermore, by the projection formula and the Leray spectral sequence, we have an isomorphism

$$H^0\big(X, \mathcal{O}_X(D) \otimes \mathcal{O}_X(-n) \otimes \pi^*\mathcal{O}_S(m)\big) \cong H^0\big(S, \pi_*\big(\mathcal{O}_X(D) \otimes \mathcal{O}_X(-n)\big) \otimes \mathcal{O}_S(m)\big).$$

In particular, the space of sections $H^0(X, \mathcal{O}_X(D) \otimes \mathcal{O}_X(-n) \otimes \pi^*\mathcal{O}_S(m))$ is nonempty. Let D' be an effective divisor defined by a section in this space.

Now, by construction,

$$\mathcal{O}_X(D') \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(-n) \otimes \pi^*\mathcal{O}_S(m),$$

and, by the choice of n, we see that the restriction to a general fibre $\mathcal{O}_F(D') \cong \mathcal{O}_F$. So $D'|_F = 0$, as D' is effective. Thus D' must be supported on fibres of $\pi\colon X \to S$.

Therefore, we have shown that any Weil divisor D may be expressed as $D \sim nH + D'$, where H is a Cartier divisor corresponding to $\mathcal{O}_X(1)$ and D' is supported on fibres of $\pi\colon X \to S$. We further note that the nH, for $n \in \mathbb{Z}$, are all distinct in $\mathrm{Cl}(X)$ and, if $n \neq 0$, also distinct from D. This is easily seen, as the restrictions of the corresponding sheaves to a general fibre do not agree.

It remains to classify the divisors supported on fibres of $\pi\colon X \to S$. Note first that, as $S \cong \mathbb{P}^1$, any two fibres of $\pi\colon X \to S$ are linearly equivalent. Furthermore, all fibres are irreducible except those of Types (II.3), (II.4), (III.3) and (III.4) in the classification of Theorem 3.2.2, which each have two components. Let F_i $(1 \leq i \leq r(X))$ denote the reducible fibres of π and denote the two components of F_i by V_i and V_i'. Note that $V_i' \sim F_i - V_i$, so any divisor supported on fibres of π can be written, up to linear equivalence, as a sum of integer multiples of F (a general fibre) and the V_i. All that remains is to show that F and the V_i are linearly independent in $\mathrm{Cl}(X)$.

So suppose that $aF + \sum_{i=1}^{r(X)} a_i V_i \sim 0$ in $\mathrm{Cl}(X)$, for some integers a and a_i. Restricting to V_i' we obtain $a_i D_i \sim 0$ in V_i', where $D_i = V_i \cap V_i'$. But this implies that $a_i = 0$ for all i. Thus we are left with $aF \sim 0$. But $\mathcal{O}_X(aF) \cong \pi^* \mathcal{O}_S(a)$, which is non-trivial if $a \neq 0$. So $a = 0$ and therefore F and the V_i are linearly independent in $\mathrm{Cl}(X)$.

Thus $\mathrm{Cl}(X)$ has rank $r(X) + 2$ and is generated over $\mathbb{Z}$ by the classes of H, F and the V_i. This completes the proof of Theorem 5.5.2. $\square$

With this in place, we have:

Corollary 5.5.3. *The rank of the second integral cohomology group of Y is given by*

$$h^2(Y, \mathbb{Z}) = r(X) + c(X) + 2$$

where $r(X)$ denotes the number of reducible fibres appearing in $\pi \colon X \to S$ and $c(X)$ denotes the number of crepant divisors calculated in Section 5.4.

Proof. Follows immediately from Theorem 5.5.2 and Proposition 5.5.1. $\square$

Bibliography

[Anc87] V. Ancona, *Vanishing and nonvanishing theorems for numerically effective line bundles on complex spaces*, Ann. Mat. Pura Appl. (4) **149** (1987), 153–164.

[Bea96] A. Beauville, *Complex algebraic surfaces*, second ed., London Math. Soc. Student Texts, vol. 34, Cambridge University Press, 1996.

[BHPvdV04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. van de Ven, *Compact complex surfaces*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Fogle, A Series of Modern Surveys in Mathematics, vol. 4, Springer-Verlag, 2004.

[BS76] C. Bănică and O. Stănăşilă, *Algebraic methods in the global theory of complex spaces*, Editura Academiei, Bucharest, 1976.

[Cai99] M. Caibăr, *Minimal models of canonical singularities and their cohomology*, Ph.D. thesis, University of Warwick, 1999.

[CP06] F. Catanese and R. Pignatelli, *Fibrations of low genus I*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 6, 1011–1049.

[Dol82] I. Dolgachev, *Weighted projective varieties*, Group Actions and Vector Fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, 1982, pp. 34–71.

[Dol96] I. V. Dolgachev, *Mirror symmetry for lattice polarised K3 surfaces*, J. Math. Sci. **81** (1996), no. 3, 2599–2630.

[FM83] R. Friedman and D. Morrison, *The birational geometry of degenerations: An overview*, The Birational Geometry of Degenerations (R. Friedman and D. Morrison, eds.), Progr. Math., vol. 29, Birkhäuser, 1983, pp. 1–32.

[FM00] O. Fujino and S. Mori, *A canonical bundle formula*, J. Differential Geom. **56** (2000), no. 1, 167–188.

[Fri83] R. Friedman, *Linear systems on anticanonical pairs*, The birational geometry of degenerations (R. Friedman and D. Morrison, eds.), Progr. Math., vol. 29, Birkhäuser, 1983, pp. 162–171.

[Fri84] ———, *A new proof of the global Torelli theorem for K3 surfaces*, Ann. of Math. (2) **120** (1984), no. 2, 237–269.

[Fuj03] O. Fujino, *A canonical bundle formula for certain algebraic fiber spaces and its applications*, Nagoya Math. J. **172** (2003), 129–171.

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977.

[Har80] ———, *Stable reflexive sheaves*, Math. Ann. **254** (1980), no. 2, 121–176.

[Hor77] E. Horikawa, *On algebraic surfaces with pencils of curves of genus 2*, Complex Analysis and Algebraic Geometry (W. L. Baily and T. Shioda, eds.), Iwanami Shoten, Tokyo, 1977, pp. 79–90.

[IF00] A. R. Iano-Fletcher, *Working with weighted complete intersections*, Explicit Birational Geometry of 3-Folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge University Press, 2000, pp. 101–173.

[Kat91] S. Katz, *Small resolutions of Gorenstein threefold singularities*, Algebraic Geometry: Sundance 1988, Contemp. Math., vol. 116, Amer. Math. Soc., 1991, pp. 61–70.

[KKMSD73] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings I*, Lecture Notes in Math., vol. 339, Springer-Verlag, 1973.

[KLM95] A. Klemm, W. Lerche, and P. Mayr, *K3-fibrations and heterotic-type II string duality*, Phys. Lett. B **357** (1995), no. 3, 313–322.

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998.

[KMM87] Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the minimal model problem*, Algebraic Geometry, Sendai, 1985 (T. Oda, ed.), Adv. Stud. in Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 283–360.

[Kol08] J. Kollár, *Exercises in the birational geometry of algebraic varieties*, Preprint, 2008, `arXiv:0809.2579`.

[KSB88] J. Kollár and N. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338.

[Kul77] V. Kulikov, *Degenerations of K3 surfaces and Enriques surfaces*, Math. USSR Izvestija **11** (1977), no. 5, 957–989.

[Kul81] ———, *On modifications of degenerations of surfaces with* $\kappa = 0$, Math. USSR Izvestija **17** (1981), no. 2, 339–342.

[Lau77] H. B. Laufer, *On minimally elliptic singularities*, Amer. J. Math. **99** (1977), no. 6, 1257–1295.

[Loo81] E. Looijenga, *Rational surfaces with an anticanonical cycle*, Ann. of Math. (2) **114** (1981), no. 2, 267–322.

[May72] A. L. Mayer, *Families of K3 surfaces*, Nagoya Math. J. **48** (1972), 1–17.

[ML89] M. Mendes-Lopes, *The relative canonical algebra for genus three fibrations*, Ph.D. thesis, University of Warwick, 1989.

[MM07] X. Ma and G. Marinescu, *Characterization of Moishezon manifolds*, Holomorphic Morse inequalities and Bergman kernels, Progr. Math., vol. 254, Birkhäuser, 2007, pp. 69–126.

[Mul09] J. P. Mullet, *On toric Calabi-Yau hypersurfaces fibered by weighted K3 hypersurfaces*, Comm. Anal. Geom. **17** (2009), no. 1, 107–138.

[Mum70] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[Nak88] N. Nakayama, *On Weierstrass models*, Algebraic Geometry and Commutative Algebra, Vol II, Kinokuniya, Tokyo, 1988, pp. 405–431.

[Per77] U. Persson, *On degenerations of algebraic surfaces*, Mem. Amer. Math. Soc. **11** (1977), no. 189.

[PP81] U. Persson and H. Pinkham, *Degenerations of surfaces with trivial canonical bundle*, Ann. of Math. (2) **113** (1981), no. 1, 45–66.

[Rei80] M. Reid, *Canonical 3-folds*, Journées de Géométrie Algébrique d'Angers, Juillet 1979 (A. Beauville, ed.), Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, pp. 273–310.

[Rei83] ———, *Minimal models of canonical* 3*-folds*, Algebraic Varieties and Analytic Varieties (Tokyo 1981), Adv. Stud. Pure Math., vol. 1, North Holland, Amsterdam, 1983, pp. 131–180.

[Rei87] ———, *Young person's guide to canonical singularities*, Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, American Mathematical Society, 1987, pp. 345–414.

[Sai74] K. Saito, *Einfach-elliptishce singularitäten*, Invent. Math. **23** (1974), 289–325.

[SB83a] N. Shepherd-Barron, *Degenerations with numerically effective canonical divisor*, The Birational Geometry of Degenerations (R. Friedman and D. Morrison, eds.), Progr. Math., vol. 29, Birkhäuser, 1983, pp. 33–84.

[SB83b] ———, *Extending polarisations on families of K3 surfaces*, The Birational Geometry of Degenerations (R. Friedman and D. Morrison, eds.), Progr. Math., vol. 29, Birkhäuser, 1983, pp. 135–161.

[Sha80] J. Shah, *A complete moduli space for K3 surfaces of degree 2*, Ann. of Math. (2) **112** (1980), no. 3, 485–510.

[Uen75] K. Ueno, *Classification theory for algebraic varieties and compact complex spaces*, Lecture Notes in Math., vol. 439, Springer-Verlag, 1975.