Models for Threefolds Fibred by K3 Surfaces of Degree Two

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This talk is based upon my current work on the explicit construction of models for threefolds fibred by K3 surfaces of degree two. Its contents may be found in more detail in the preprint [Tho11a] and in my doctoral thesis [Tho11b], a copy of which is currently available on my website:

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We begin by defining our objects of study:

Definition 1 *A* K3 surface of degree two *is a nonsingular projective surface X satisfying $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$, along with an ample invertible sheaf $\mathcal{L}$ on X that has self-intersection number $\mathcal{L}.\mathcal{L} = 2$.*

A Riemann-Roch calculation shows that the invertible sheaf $\mathcal{L}$ defines an embedding of X as a sextic hypersurface in weighted projective space $X \cong X_6 \subset \mathbb{P}(1,1,1,3)$. Equivalently, we may see X as a double cover $X \to \mathbb{P}^2$ ramified over a smooth sextic curve.

We are interested in studying threefolds that admit fibrations by such surfaces. Formally we define:

Definition 2 *Let S be a nonsingular complex curve. A* threefold fibred by K3 surfaces of degree two *over S is a triple $(X, \pi, \mathcal{L})$ satisfying:*

- *X is a nonsingular 3-dimensional complex variety;*
- *$\pi \colon X \to S$ is a projective, flat, surjective morphism with connected fibres, whose general fibres are K3 surfaces;*
- *$\mathcal{L}$ is an invertible sheaf on X which induces an ample invertible sheaf $\mathcal{L}_s$ on a general fibre X_s, that has self-intersection number $\mathcal{L}_s.\mathcal{L}_s = 2$.*

In order to study such threefolds, we would like to find explicit birational models for them. As K3 surfaces may be seen as a 2-dimensional analogue of elliptic curves, it makes sense to start by drawing inspiration from the well-developed theory of elliptic fibrations. In particular, we will attempt to emulate Nakayama's [Nak88] construction of the Weierstrass model for an elliptic fibration with a section.

In order to do this we use the fact that a K3 surface of degree two may be embedded as a sextic hypersurface in $\mathbb{P}(1,1,1,3)$, and attempt to construct a model for our threefold fibred by K3 surfaces of degree two $(X, \pi, \mathcal{L})$ inside a bundle of weighted projective spaces $\mathbb{P}(1,1,1,3)$ on the base curve S.

To construct this bundle, we start with a rank 3 vector bundle $\mathcal{E}_1$ and a line bundle $\mathcal{E}_3^+$ on S, then take the relative **Proj** of the weighted symmetric algebra that has $\mathcal{E}_1$ in degree 1 and $\mathcal{E}_3^+$ in degree 3. But how do we define $\mathcal{E}_1$ and $\mathcal{E}_3^+$?

$\mathcal{E}_1$ is easily defined, we simply set $\mathcal{E}_1 := \pi_*\mathcal{L}$. We would like to set $\mathcal{E}_3^+$ to be the cokernel of the map $\sigma_3\colon \mathrm{Sym}^3(\mathcal{E}_1) \to \pi_*\mathcal{L}^{\otimes 3}$, but unfortunately this cokernel is not necessarily locally free. Instead, we define $\mathcal{E}_3^+$ to be the reflexivisation

$$\mathcal{E}_3^+ := \left(\mathrm{coker}(\sigma_3\colon \mathrm{Sym}^3(\mathcal{E}_1) \longrightarrow \pi_*\mathcal{L}^{\otimes 3})\right)^{\vee\vee}.$$

Finally, we take a sextic hypersurface in this bundle to get a model W for X, called the *K3-Weierstrass model*. The model W admits a natural projection $p\colon W \to S$ and may be seen as a double cover of the $\mathbb{P}^2$-bundle $\mathbb{P}_S(\mathcal{E}_1)$.

Theorem 3 [Tho11b, Theorem 1.3.3]. *There is a birational map $\mu\colon X \dashrightarrow W$ over S that is an isomorphism on the general fibre.*

Note that this theorem tells us nothing about the special fibres. To see what happens to them, we need to understand what such fibres can look like.

Example 4 We can obtain one type of special fibre by relaxing the definition of a K3 surface of degree two. Let X be a K3 surface and let $\mathcal{L}$ be a line bundle on X that is nef and big (but not ample) with self-intersection number $\mathcal{L}.\mathcal{L} = 2$. Then $\mathcal{L}$ is not necessarily generated by its global sections. If this is the case, a Riemann-Roch calculation shows that X admits a birational morphism to a complete intersection $X \to X_{2,6} \subset \mathbb{P}(1,1,1,2,3)$, where the degree two relation does no involve the degree two variable. In this case, we see that $X_{2,6}$ cannot be seen as a double cover of $\mathbb{P}^2$. Instead, it is a double cover of a quadric cone $X_2 \subset \mathbb{P}(1,1,1,2)$, ramified over a smooth sextic and the vertex $(0:0:0:1)$. Such a K3 surface is called *unigonal*.

Returning to the K3-Weierstrass model W, we see that the fibres of $p\colon W \to S$ all admit morphisms to $\mathbb{P}^2$. So something must happen to the unigonal fibres. In reality, they are destroyed and completely replaced by the birational map $\mu\colon X - \to W$, a process that can make W highly singular. This is a big problem with the model W, and means that it is not of much use for studying the properties of X.

In fact, it can be shown that, under certain assumptions, the unigonal fibres form the only obstruction to the construction of a good model for X. We have:

Theorem 5 [Tho10, Theorem 4.1]. *Let $\pi\colon X \to \Delta := \{z \in \mathbb{C} : 0 \leq |z| < 1\}$ be a semistable (i.e. the central fibre is reduced and has normal crossings) degeneration of K3 surfaces with $\omega_X \cong \mathcal{O}_X$ and let H be a divisor on X that is effective, nef and flat over Δ, that induces an ample divisor H_s with $H_s.H_s = 2$ on a general fibre. Then H defines a morphism that maps the central fibre to either:*

- *A sextic hypersurface $X_6 \subset \mathbb{P}(1,1,1,3)$ (i.e. a double cover of $\mathbb{P}^2$ ramified over a sextic curve); or*
- *A complete intersection $X_{2,6} \subset \mathbb{P}(1,1,1,2,3)$, where the degree two relation does not involve the degree two variable (i.e. a double cover of a quadric cone, ramified over a sextic curve and the vertex of the cone).*

Note that these surfaces may be singular (even non-normal).

Following an idea of Catanese and Pignatelli [CP06], we use this to refine our construction. Instead of a $\mathbb{P}^2$-bundle, we construct a bundle of rational surfaces on S, which we allow to degenerate to quadric cones. Taking a double cover of this bundle, we obtain a new model $\pi^c\colon X^c \to S$. As the fibres in this model are no longer forced to admit a morphism to $\mathbb{P}^2$, this model behaves much better on the unigonal fibres.

In order to construct this model, we need to alter some of the assumptions on $(X, \pi, \mathcal{L})$ that we made originally. This will allow us to use the explicit description of the fibres obtained from Theorem 5. We assume:

- The polarisation $\mathcal{L}$ is locally flat, i.e. for all $s \in S$ there exists an open set $U_s \ni s$ and a section in $\Gamma(\pi^{-1}(U_s), \mathcal{L})$ that defines an effective and flat divisor over U_s;
- X is allowed to have Gorenstein terminal singularities;

- There exists an analytic resolution $f\colon Y \to X$ of X such that Y is semistable and the exceptional locus of f has codimension 2 in Y.

Under these new assumptions, the model that we would like to explicitly construct is the *relative log canonical model* of $(X, \pi, \mathcal{L})$, defined as:

$$X^c := \mathbf{Proj}_S\big(\bigoplus_{n\geq 0} \pi_*(\omega_X^{\otimes n} \otimes \mathcal{L}^{\otimes n})\big).$$

This model has been widely studied in relation to the minimal model program. In particular, it has the following desirable properties:

- There exists a birational map $\phi\colon X - \to X^c$ over S;
- The exceptional set of ϕ^{-1} has codimension 2 in X^c (so ϕ cannot "destroy and replace" any fibres);
- X^c has at worst canonical singularities (so the singularities of X^c are controlled).

In order to construct this model explicitly, we study the sheaf of $\mathcal{O}_S$-algebras

$$\mathcal{R} := \bigoplus_{n\geq 0} \pi_*(\omega_X^{\otimes n} \otimes \mathcal{L}^{\otimes n}),$$

called the *relative log canonical algebra.*

Let $\mathcal{A}$ be the graded subalgebra of $\mathcal{R}$ generated in degrees 1 and 2. Then we have:

Lemma 6 *The inclusion $\mathcal{A} \subset \mathcal{R}$ yields a double cover*

$$\psi\colon X^c := \mathbf{Proj}_S\mathcal{R} \longrightarrow \mathbf{Proj}_S\mathcal{A}.$$

Note that $\mathbf{Proj}_S\mathcal{A}$ is the fibration of S by rational surfaces that we mentioned earlier.

This lemma means that we can use the algebra $\mathcal{A}$ as a "stepping stone" on the way to the construction of $\mathcal{R}$. So we begin by constructing $\mathcal{A}$. By definition, the graded parts

$$\mathcal{A}_1 \cong \pi_*(\omega_X \otimes \mathcal{L}) := \mathcal{E}_1,$$
$$\mathcal{A}_2 \cong \pi_*(\omega_X^{\otimes 2} \otimes \mathcal{L}^{\otimes 2}) := \mathcal{E}_2,$$

and we have an exact sequence induced by multiplication

$$(*) \qquad 0 \longrightarrow \mathrm{Sym}^2(\mathcal{E}_1) \xrightarrow{\sigma_2} \mathcal{E}_2 \longrightarrow \mathcal{T}_2 \longrightarrow 0.$$

Furthermore, we have

Lemma 7 [Tho11a, Lemma 5.1] *$\mathcal{T}_2 \cong \mathcal{O}_\tau$ for some effective divisor τ on S that is supported on the points of S corresponding to the unigonal fibres of $\pi\colon X \to S$.*

and

Proposition 8 [Tho11a, Proposition 5.4] *$\mathcal{A}$ can be completely reconstructed from the data $\mathcal{E}_1$, τ and the extension of $\mathcal{O}_\tau$ by $\mathrm{Sym}^2(\mathcal{E}_1)$ given by the exact sequence $(*)$.*

Given this, we just need to find a way to reconstruct $\mathcal{R}$ from $\mathcal{A}$. We begin with:

Proposition 9 [Tho11a, Proposition 5.6] *Let $\mathcal{E}_3^+$ denote the reflexivisation*

$$\mathcal{E}_3^+ := \left(\mathrm{coker}(\sigma_3\colon \mathrm{Sym}^3(\mathcal{E}_1) \longrightarrow \pi_*(\omega_X^{\otimes 3} \otimes \mathcal{L}^{\otimes 3}))\right)^{\vee\vee}.$$

Then $\mathcal{R} \cong \mathcal{A} \oplus (\mathcal{A}[-3] \otimes \mathcal{E}_3^+)$ as a graded $\mathcal{A}$-module.

This proposition gives $\mathcal{R}$ the structure of an $\mathcal{A}$-module, but to fully reconstruct it we need to equip it with a multiplicative structure (i.e. the structure of an $\mathcal{A}$-algebra). Such a multiplicative structure is completely determined by a map $\beta\colon (\mathcal{E}_3^+)^2 \to \mathcal{A}_6$.

This information is enough to determine the branch locus of the double cover $\psi\colon \mathbf{Proj}_S\mathcal{R} \to \mathbf{Proj}_S\mathcal{A}$. To calculate this locus, begin by considering a point P in the support of τ. Then the fibre of $\mathbf{Proj}_S\mathcal{A}$ over P is a quadric cone $X_2 \subset \mathbb{P}(1,1,1,2)$, where the degree two relation does not involve the degree two variable, and is singular at the point $(0:0:0:1)$. Taking all such singular points associated to all of the points in $\mathrm{Supp}(\tau)$, we get a subset of $\mathbf{Proj}_S\mathcal{A}$ that will be denoted by $\mathcal{P}$. Then the branch locus of ψ consists of the set of isolated points $\mathcal{P}$ together with the divisor $B_\mathcal{A}$ in the linear system $|\mathcal{O}_{\mathbf{Proj}_S\mathcal{A}}(6) \otimes \pi_\mathcal{A}^*(\mathcal{E}_3^+)^{-2}|$ determined by the map β.

Thus, we have a 5-tuple of data on S:

- $\mathcal{E}_1$ is a rank 3 vector bundle on S;
- τ is an effective divisor on S;
- $\xi \in \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_\tau, \mathrm{Sym}^2(\mathcal{E}_1))/\mathrm{Aut}_{\mathcal{O}_S}(\mathcal{O}_\tau)$ yields an extension of $\mathcal{O}_\tau$ by $\mathrm{Sym}^2(\mathcal{E}_1)$, giving a vector bundle $\mathcal{E}_2$ and a map $\sigma_2\colon \mathrm{Sym}^2(\mathcal{E}_1) \to \mathcal{E}_2$;
- $\mathcal{E}_3^+$ is a vector bundle on S;

- $\beta \in \mathbb{P}(H^0(S, \mathcal{A}_6 \otimes (\mathcal{E}_3^+)^{-2}))$, where $\mathcal{A}_6$ is the degree six part of the algebra $\mathcal{A}$ defined using $\mathcal{E}_1$, $\mathcal{E}_2$ and σ.

Finally, we have:

Theorem 10 [Tho11a, Theorem 6.2] *Any threefold fibred by K3 surfaces of degree two uniquely determines a* 5*-tuple as above, from which its relative log canonical model can be explicitly reconstructed.*

Furthermore, let $\mathcal{R}$ be an $\mathcal{O}_S$-algebra constructed from a 5*-tuple as above, and assume:*

- *If $B_{\mathcal{A}}$ is the divisor of β on $\mathbf{Proj}_S(\mathcal{A})$, then $B_{\mathcal{A}}$ contains no points from the set $\mathcal{P}$,*

- *$\mathbf{Proj}_S(\mathcal{R})$ has canonical singularities and has a semistable analytic resolution that modifies only finitely many fibres,*

then there exists a threefold fibred by K3 surfaces of degree two that has relative log canonical model given by $\mathbf{Proj}_S(\mathcal{R})$.

References

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