The Mirror Abel-Jacobi Map

(Joint work in progress with C. Doran & M. Kerr)

Classical Theory.

E elliptic $/\mathbb{C}$

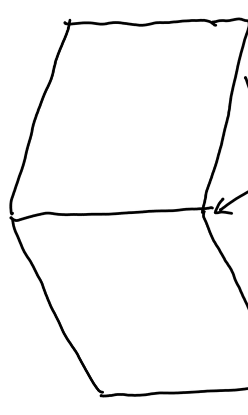
$$0 \to H^1(E,\mathbb{Z}) \to H^1(E,\mathcal{O}_E) \to \underset{\overset{\|}{\mathrm{Pic}(E)}}{H^1(E,\mathcal{O}_E^*)} \xrightarrow{\deg} \underset{\overset{\|}{\mathbb{Z}}}{H^2(E,\mathbb{Z})}$$

$$\frac{H^1(E,\mathcal{O}_E)}{H^1(E,\mathbb{Z})} \xrightarrow{\sim} \mathrm{Ker}(\deg) = \mathrm{Pic}^0(E)$$

Inverse:

$$AJ: \mathrm{Pic}^0(E) \to \frac{H^1(E,\mathcal{O}_E)}{H^1(E,\mathbb{Z})} \cong \frac{H^0(E,\Omega^1_E)^\vee}{H_1(E,\mathbb{Z})} \cong \underset{\underset{E}{\wr\wr}}{\frac{\mathbb{C}}{\mathbb{Z}\oplus\mathbb{Z}\rho}}$$ for ρ period pt of E

Tyurin Surfaces.

$X_0 =$ 

V_1 rational

E smooth elliptic, $E \in |-K_{V_i}|$, $K_{V_1}^2 + K_{V_2}^2 = 0$

V_2 rational

$$H^2(V_1,\mathbb{Z}) \oplus H^2(V_2,\mathbb{Z}) = \mathrm{Pic}(V_1) \oplus \mathrm{Pic}(V_2) \xrightarrow{\text{restrict}} \mathrm{Pic}(E)$$

$\xi = (E,-E) \in H^2(V_1) \oplus H^2(V_2)$

$(\alpha_1,\alpha_2) \in \xi^\perp \Rightarrow \alpha_1.E = \alpha_2.E_2 \Rightarrow$ restriction is in $\mathrm{Pic}^0(E)$

Compose with AJ

$$\psi: \xi^\perp \longrightarrow \frac{H^0(E,\Omega^1_E)^\vee}{H_1(E,\mathbb{Z})} \cong E$$

Properties:

1) $\psi(\xi) = 0$ iff X_0 admits a smoothing to a K3 [Friedman]

2) If this holds & $\psi(\alpha) = 0$, then the smoothing can be chosen s.t α becomes the a class of a divisor on K3

Moduli of K3

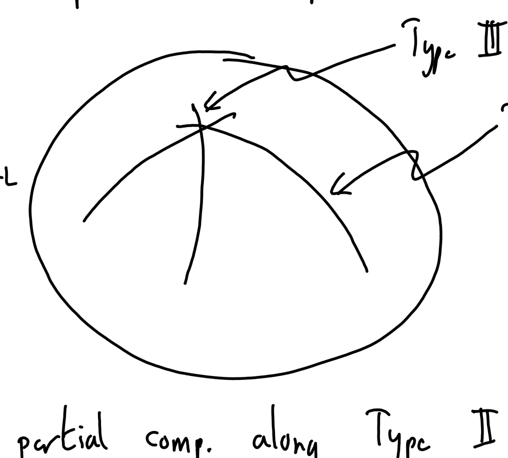
X K3

$H^2(X,\mathbb{Z}) \cong \Lambda_{K3}$ (rk 22 lattice)

$L \subset \Lambda_{K3}$ be sub. of sign $(1,n)$

L-polarisation of X is embedding $L \hookrightarrow \mathrm{Pic}(X)$

Moduli space of L-pol. K3, $(19-n)$-dim'l:

$\mathcal{M}_L$ 

Type III bdry, correspond to isotropic vectors.

Type II bdry, correspond to rk 2 isotropic sublattices in $L^\perp$

$\exists$ partial comp. along Type II bdries s.t points correspond to Tyurin surfaces, moduli given by AJ maps. [Friedman, Alexeev-Engel]

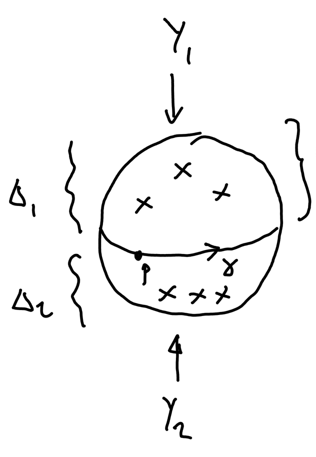
Mirror:

$\pi: Y \to \mathbb{P}^1$ ell. fibred K3

$\omega \in H^2(Y,\mathbb{R})$ symplectic form, $B \in H^2(Y,\mathbb{R})/H^2(Y,\mathbb{Z})$

$\gamma \subset \mathbb{P}^1$ simple oriented loop avoids critical locus

$p \in \gamma$ point, F fibre over p.



$e =$ Euler # of fibres enclosed by γ in a/c sense.

$$\begin{array}{ccccc} Y & = & Y_1 & \cup & Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & = & \Delta_1 & \underset{\gamma}{\cup} & \Delta_2 \end{array}$$

Assume (*) $\exists$ symplectic basis (a,b) for $H_1(F,\mathbb{Z})$

1) Monodromy around γ is

$$\begin{array}{ccc} & a & b \\ a & 1 & e-12 \\ b & 0 & 1 \end{array}$$

2) Class τ obtained by parallel transporting a around γ is primitive in $H^2(Y,\mathbb{Z})$

$[H^0(E,\Omega^1_E)^\vee / H_1(E,\mathbb{Z})]$

$H^0(E,\Omega^1_E)^\vee \overset{\text{mirror}}{\rightsquigarrow} H^0(F,\mathbb{C})^\vee$

$H_1(E,\mathbb{Z}) \xrightarrow{\text{mirror}} H^{\text{even}}_{\text{quantum}}(F,\mathbb{Z})$

$\uparrow$ Free $\mathbb{Z}$-mod in $H^{\text{even}}(F,\mathbb{C})$ gen. by $[pt]$ & $[F] + \rho[pt]$ for $\rho = \int_F (B + i\omega)$

$$\begin{aligned} H^{\text{even}}_{\text{quantum}}(F,\mathbb{Z}) &\hookrightarrow H^0(F,\mathbb{C})^\vee \cong \mathbb{C} \\ \alpha &\longmapsto \alpha \cup [F] \\ [pt] &\longmapsto 1 \\ [F] + \rho[pt] &\longmapsto \rho \end{aligned}$$

$$\frac{H^0(F,\mathbb{C})^\vee}{H^{\text{even}}_{\text{quantum}}(F,\mathbb{Z})} \cong \frac{\mathbb{C}}{\mathbb{Z}\oplus\mathbb{Z}\rho}$$

Mirror to deg:

$$\phi: H_2(Y_1,F) \oplus H_2(Y_2,F) \xrightarrow[\text{boundaries}]{\partial_1 - \partial_2} H_1(F,\mathbb{Z})$$

Mirror AJ: [AKO]

$$\check{\psi}: \mathrm{Ker}(\phi) \longrightarrow \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\rho$$

For $\alpha \in \mathrm{Ker}(\phi)$, let $\bar{\alpha}$ be lift to $H^2(Y,\mathbb{Z})$,

$$\check{\psi}(\alpha) = \int_{\bar{\alpha}} (B + i\omega)$$

K3 mirror symmetry.

$L \subset \Lambda_{K3}$, $I \subset L^\perp$ rk 2 isotropic

$e \in I$ m-admissible vector $\rightsquigarrow L^\perp = \underset{\hookrightarrow \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}}{\overset{e}{\overset{\cap}{H(m)}}} \oplus \check{L}$

$\check{L}$ mirror lattice to L. [Dolgachev]

Assume chosen s.t we can identify:

$$\begin{aligned} \Lambda_{K3} &\xrightarrow{\sim} H^2(Y,\mathbb{Z}) \\ \check{L} &\hookrightarrow NS(Y) \text{ s.t } [F] \in \check{L} \text{ and } \tau \in \check{L}^\perp \\ e &\longmapsto \tau \\ I &\longrightarrow \mathrm{Span}(\tau,[F]) \end{aligned}$$

Assume $\omega \in \check{L} \otimes \mathbb{R} \subset H^2(Y,\mathbb{R})$ Kähler form

$$B \in \frac{\check{L}\otimes\mathbb{R}}{\check{L}} \subset \frac{H^2(Y,\mathbb{R})}{H^2(Y,\mathbb{Z})}$$

Thm: $\exists$ a degeneration $X \rightsquigarrow X_0$ of L-polarised K3's to a Tyurin surface X_0 s.t the AJ map for X_0 coincides withe the mirror AJ map defined by $B + i\omega$.

Questions

1) Suppose we have fibrations Y_1, Y_2 over discs equipped with $B_1 + i\omega_1$ & $B_2 + i\omega_2$ satisfying (*) s.t

- $e(Y_1) + e(Y_2) = 24$
- $\int_F B_1 + i\omega_1 = \int_F B_2 + i\omega_2 =: \rho$
- $\check{\psi}(\tau) = 0$
- If $\zeta_i \in H_2(Y_i,F)$ is class obtained by parallel transport of b around γ, then $\check{\psi}(\zeta_1 - \zeta_2) = 0$

Then Y_1 & Y_2 glue to smooth K3 equipped with $B + i\omega$

2) Suppose further that $\check{\psi}(\alpha) = 0$ for $\alpha \in \mathrm{Ker}(\phi)$, then gluing can be chosen s.t α gives a transcendental class on Y.

$$X \rightsquigarrow V_1 \cup_E V_2 \overset{\text{mirror}}{\rightsquigarrow} Y_1 \cup Y_2 \overset{\text{glue}}{\rightsquigarrow} Y.$$

mirror.