Variations of Hodge Structure and the Period Map

Alan Thompson

11th November 2013.
Concentrated Graduate Course preceeding Workshop 4 on Hodge Theory in String Theory.
Fields Institute, Toronto.

Abstract: We begin by defining the period map, which relates families of Kähler manifolds to the families of Hodge structures defined on their cohomology, and discuss its properties. This will lead us to the more general definition of a variation of Hodge structure and the Gauss-Manin connection.

This talk is based upon an introductory paper by Griffiths [GT84] and Chapter 4 of the book by Carlson, Müller-Stach and Peters [CMSP03]. References to specific results and any material not contained in these two sources will be provided where appropriate.

1 Polarised Hodge structures.

Let X be a compact Kähler manifold with Kähler form ω. Fix once and for all an integer $n \geq 0$. Let $H_{\mathbb{Z}} := H^n(X, \mathbb{Z})/\text{torsion}$ and $H := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \cong H^n(X, \mathbb{C})$.

Recall that we may decompose H as a direct sum

$$H = \bigoplus_{p+q=n} H^{p,q}.$$

The data $(H_{\mathbb{Z}}, H^{p,q})$ is a pure Hodge structure of weight n.

Remark 1. In this talk, by a *pure Hodge structure of weight* n we will mean a finitely generated free abelian group $H_{\mathbb{Z}}$ along with a decomposition $H = \bigoplus_{p+q=n} H^{p,q}$ of $H := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying $H^{p,q} = \overline{H^{q,p}}$.

Remark 2. It is also common to see a pure Hodge structure of weight n defined by a decreasing filtration $\{F^p\}$ on H

$$H = F^0 \supset F^1 \supset \cdots \supset F^n$$

such that $H \cong F^p \oplus \overline{F^{n-p+1}}$. The two definitions are completely equivalent: given a decomposition $H = \bigoplus_{p+q=n} H^{p,q}$ we may define a filtration by setting $F^p := H^{n,0} \oplus \cdots \oplus H^{p,n-p}$, and given a filtration $\{F^p\}$, we may define a decomposition by setting $H^{p,q} := F^p \cap \overline{F^q}$.

We can use ω to define a nondegenerate bilinear form $Q\colon H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ by

$$Q(\xi,\eta) := (-1)^{\frac{n(n-1)}{2}} \int_X \xi \wedge \eta \wedge \omega^{\dim(X)-n}.$$

Q extends to H by linearity and has the following properties:

(1) Q is symmetric if n is even and skew-symmetric if n is odd.

(2) $Q(\xi,\eta) = 0$ for $\xi \in H^{p,q}$ and $\eta \in H^{p',q'}$ with $p \neq q'$.

(3) $i^{p-q}Q(\xi,\bar{\xi}) > 0$ for $\xi \in H^{p,q}$ non-zero.

Conditions (2) and (3) are called the *Hodge-Riemann bilinear relations.*

This motivates the following definition:

Definition 1. A *polarised Hodge structure of weight* n consists of a pure Hodge structure $(H_{\mathbb{Z}}, H^{p,q})$ of weight n together with a nondegenerate integer bilinear form Q on $H_{\mathbb{Z}}$ which extends to H by linearity and satisfies (1)-(3) above.

Remark 3. It is also common to see the Hodge-Riemann bilinear relations written in terms of the filtration $\{F^p\}$. In this case they become:

(2') $Q(F^p, F^{n-p+1}) = 0$.

(3') $Q(C\xi,\bar{\xi}) > 0$ for any nonzero $\xi \in H$, where $C\colon H \to H$ is the *Weil operator* defined by $C|_{H^{p,q}} = i^{p-q}$.

2 The local period mapping.

Let $f\colon \mathcal{X} \to \Delta$ be a proper smooth surjective morphism onto a complex polydisc Δ, whose fibres X_b are compact Kähler varieties for all $b \in \Delta$. Assume that there exists $\omega \in H^2(\mathcal{X},\mathbb{Z})$ such that, for each $b \in \Delta$, the restriction $\omega_b := \omega|_{X_b}$ is a Kähler class.

As f is smooth and Δ is simply connected, there is a *unique* isomorphism $H^n(X_b,\mathbb{Z}) \cong H^n(X_{b'},\mathbb{Z})$ for any $b, b' \in \Delta$. We may therefore define $H_{\mathbb{Z}} := H^n(X_b,\mathbb{Z})/\text{torsion}$ and $H := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \cong H^n(X_b,\mathbb{C})$, and these definitions do not depend upon the choice of $b \in \Delta$. The class ω induces a bilinear form Q on $H_{\mathbb{Z}}$ as above, which extends to H by linearity.

The induced isomorphisms $H^n(X_b,\mathbb{C}) \cong H^n(X_{b'},\mathbb{C})$ do not preserve the Hodge decompositions of these spaces; instead, the Hodge decomposition of $H^n(X_b,\mathbb{C})$ varies continuously with $b \in \Delta$. In particular, the dimensions $h^{p,q} := \dim(H^{p,q})$ of the subspaces are fixed (these dimensions are called the *Hodge numbers*).

Furthermore, if $\{F_b^p\}$ is the Hodge filtration on $H^n(X_b,\mathbb{C})$, we find that $\{F_b^p\}$ has the following properties:

$$\frac{\partial F_b^p}{\partial \bar{b}} \subset F_b^p \qquad (\textit{holomorphicity})$$

$$\frac{\partial F_b^p}{\partial b} \subset F_b^{p-1} \qquad (\textit{Griffiths transversality})$$

Define:

Definition 2. Let $\mathcal{D}$ denote the set of all collections of subspaces $\{H^{p,q}\}$ of H such that $H = \bigoplus_{p+q=n} H^{p,q}$ and $\dim(H^{p,q}) = h^{p,q}$, on which Q satisfies the Hodge-Riemann bilinear relations (2) and (3).

Remark 4. In terms of filtrations, $\mathcal{D}$ may be defined as the set of all filtrations

$$H = F^0 \supset F^1 \supset \cdots \supset F^n,$$

with $\dim(F^p) = h^{n,0} + \cdots + h^{p,n-p}$, on which Q satisfies (2') and (3').

There is a well-defined morphism $\phi\colon \Delta \to \mathcal{D}$, where ϕ takes $b \in \Delta$ to the point in $\mathcal{D}$ corresponding to the Hodge decomposition of $H^n(X_b, \mathbb{C})$.

This morphism has the following properties:

- ϕ is holomorphic and is called the *local period mapping* (this property is equivalent to the holomorphicity property of the filtration $\{F_b^p\}$).
- $\mathcal{D}$ is a complex manifold, called the *local period domain*.

3 The global period mapping.

Studying families over a polydisc Δ does not allow us to consider the situation where we have a family over a base that is not topologically trivial. If our base B is a more general complex manifold, then the isomorphism $H^n(X_b, \mathbb{Z}) \cong H^n(X_{b'}, \mathbb{Z})$ for $b, b' \in B$ is not necessarily unique. This means that the period map is no longer well-defined. To compensate for this, we must quotient the period domain $\mathcal{D}$ by the action of monodromy.

Let

$$G := \{g\colon H_{\mathbb{Z}} \to H_{\mathbb{Z}} \mid Q(g\xi, g\eta) = Q(\xi, \eta) \text{ for all } \xi, \eta \in H_{\mathbb{Z}}\}. \tag{1}$$

G acts on $\mathcal{D}$ in the obvious way. We have a monodromy representation

$$\rho\colon \pi_1(B) \to G.$$

Suppose that $\Gamma \subset G$ contains the image of ρ. Then we have a well-defined map $\phi\colon B \to \Gamma\backslash\mathcal{D}$. This is the *global period mapping.* The quotient $\Gamma\backslash\mathcal{D}$ is called the *period domain.*

4 Variation of polarised Hodge structure.

We can use this to define abstract variations of polarised Hodge structure.

Let $H_{\mathbb{Z}}$ be a finitely generated free abelian group equipped with a nondegenerate bilinear form Q. Let $\mathcal{D}$ be a local period domain classifying Hodge structures of weight n on $H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ polarised by the bilinear form Q, with given Hodge numbers $\{h^{p,q}\}$ (defined as in Definition 2). Define the group G as in Equation (1) and let $\Gamma \subset G$ be a subgroup. Finally, let B be a complex manifold.

Definition 3. A map $\phi\colon B \to \Gamma\backslash\mathcal{D}$ defines a *polarised variation of Hodge structures* on B if

(i) for every point $b \in B$, the map ϕ restricted to a small polydisc around b lifts to a holomorphic map $\tilde{\phi}\colon \Delta \to \mathcal{D}$ (ϕ is said to be *locally liftable*), and

(ii) the local lifts at any point satisfy Griffiths transversality.

5 Variations of Hodge structure.

In more generality, let B be a complex manifold and let $E_{\mathbb{Z}}$ be a locally constant system of finitely generated free $\mathbb{Z}$-modules on B. Define $E := E_{\mathbb{Z}} \otimes \mathcal{O}_B$. Then E is a complex vector bundle which carries a natural flat connection $\nabla\colon E \to E \otimes \Omega^1_B$ (the *Gauss-Manin connection*), induced by $d\colon \mathcal{O}_B \to \Omega^1_B$. Let $\{F^p\}$ be a filtration on E by holomorphic sub-bundles.

Definition 4. This data defines a *variation of Hodge strucures* on B if

(i) $\{F^p\}$ induces Hodge structures on the fibres of E, and

(ii) If s is a section of F^p and ζ is a vector field of type $(1,0)$, then $\nabla_\zeta s$ is a section of F^{p-1} (this is an alternative formulation of Griffiths transversality).

If $E_{\mathbb{Z}}$ carries a nondegenerate bilinear form $Q\colon E_{\mathbb{Z}} \times E_{\mathbb{Z}} \to \mathbb{Z}$, we have a *polarised variation of Hodge structures* if, additionally:

(iii) Q defines a polarised Hodge structure on the fibres of E, and

(iv) Q is flat with respect to ∇, i.e.

$$dQ(s, s') = Q(\nabla s, s') + Q(s, \nabla s')$$

for any sections s, s' of E.

Lemma 5. [CMSP03, Lemma 4.5.3] *This definition agrees with the previous one.*

References

[CMSP03] J. Carlson, S. Müller-Stach, and C. Peters, *Period mappings and period domains*, Cambridge Studies in Advanced Mathematics, vol. 85, Cambridge University Press, 2003.

[GT84] P. Griffiths and L. Tu, *Variation of Hodge structure*, Topics in Transcendental Algebraic Geometry (Princeton, N.J., 1981/1982) (P. Griffiths, ed.), Ann. of Math. Stud., vol. 106, Princeton Univ. Press, 1984, pp. 3–28.