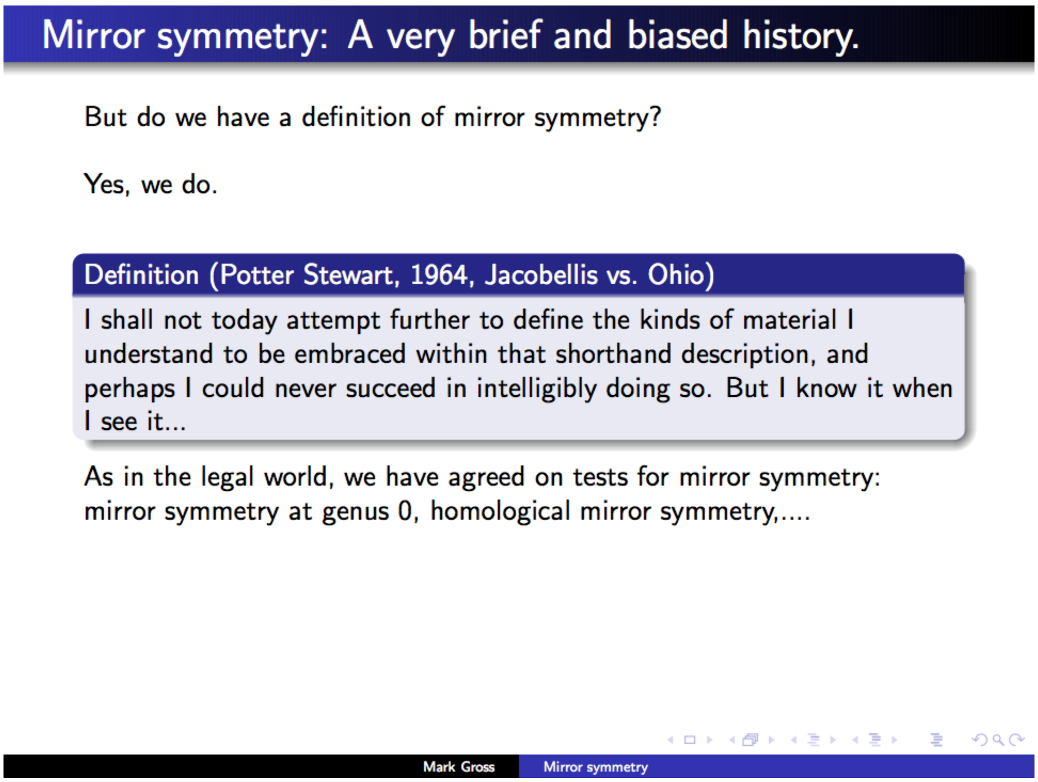
FIRST GW INVARIANTS

Thanks to Mark Gross, [7, Slide 16], we have a definition of mirror symmetry:



Suggested reading

- Probably the best preparation is to start reading [6].
- Read about *Semi-flat SYZ mirror symmetry*. References: [3, Section 6.2], the original paper [10] or various introduction by various people (ask google).
- Understand how mirror symmetry works for elliptic curves, by e.g. reading [1, Example 2.4]. Note how mirror symmetry exchanges the symplectic and complex geometry.
- Watch the talk http://media.kias.re.kr/detailPage.do?pro_seq=739&type=p on the KIAS media archive given by Helge Ruddat with title *Introduction to the Gross-Siebert program.*

The lecture series will be devoted to mirror symmetry, though not this exercise session. Throughout mirror symmetry, Gromov-Witten invariants play a crucial role. In fact, the initial way mirror symmetry caught the attention of algebraic geometers was through its ability for enumerative predictions in terms of Gromov-Witten invariants. While the general theory is rather intricate, basic examples of Gromov-Witten invariants can be calculated 'by hand'. The goal of this exercise session is for the reader to acquire familiarity with the concepts surrounding curve counting theories.

Here, our target variety is the projective plane $\mathbb{P}^2$. Denote by N_d the number of rational degree d curves passing through $3d-1$ *general points*. N_d is your first Gromov-Witten invariant. General here means that no 3 points lie on a line, no 6 on a conic etc. First we justify (though not prove fully) in two exercises why N_d is a finite number.

Exercise 1. For $g > 1$, consider the moduli space $\mathcal{M}_g$ of isomorphism classes of smooth curves of genus g.[1] In this exercise, we calculate the dimension of $\mathcal{M}_g$ by probing covers over $\mathbb{P}^1$. This is [2, Exercise 23.2.1].

(1) Let C be a smooth genus g curve. Then C has a g-dimensional family of line bundles of each degree d. Let L be such a line bundle. For large d, $\mathrm{H}^1(C, L) = 0$ and Riemann-Roch tells us that $\dim \mathrm{H}^0(C, L) = d - g + 1$. Choosing two general sections $s_0, s_1 \in \mathrm{H}^0(C, L)$ yields a cover $C \to \mathbb{P}^1$ of degree d. Compute the dimension of such covers in terms of $\dim \mathcal{M}_g$.

(2) Next, Riemann-Hurwitz tells us that a general cover $C \to \mathbb{P}^1$ has $2d + 2g - 2$ branch points. Hence the dimension of such covers (up to isomorphism) is $2d + 2g - 2$, corresponding to the independent motions of the branch points. Conclude that $\dim \mathcal{M}_g = 3g - 3$.

There is a *stacky* sense in which $\dim \mathcal{M}_g = 3g-3$ also holds for $g = 0, 1$. In particular, $\dim \mathcal{M}_0 = -3$ can be understood as the group of automorphisms of $\mathbb{P}^1$ being 3-dimensional. What is this group?

The goal of the next exercise is to justify that N_d is a finite number. We will argue that the space of degree d rational curves in $\mathbb{P}^2$ is $3d-1$ dimensional. Moreover, requiring that a curve meets a general point cuts down the dimension by 1. Hence requiring that a degree d curve meets $3d - 1$ general points yields a zero-dimensional moduli problem.

Exercise 2. (This is a subset of section $\frac{1}{2}$ and section $1\frac{1}{2}$, *Deformation theory* of [11]. See also section 24.4 of [2].) We make some idealized assumptions. Assume that $C \subset X$ is a smooth curve inside a smooth projective

[1] More precisely, $\mathcal{M}_g$ is a smooth orbifold when $g > 2$, a smooth Deligne-Mumford stack when $g = 2$, and an Artin stack when $g = 0, 1$. The 0-dimensional $\mathcal{M}_1$ should not be confused with the (1-dimensional) moduli space of elliptic curves $\mathcal{M}_{1,1}$, which consists of a smooth genus 1 curve and a marked point regarded as the identity for the group law.

variety (that's already not our case unless $d = 1, 2$). Then infinitesimal deformations of C are given by global sections of the normal bundle ν_C and thus lie in $H^0(C, \nu_C)$, while obstructions to deformations lie in $H^1(C, \nu_C)$.[2] This means that a deformation $s \in H^0(C, \nu_C)$ will be admissible if some class $ob(s) \in H^1(C, \nu_C)$ vanishes. Given that these are vector spaces, the *expected*[3] dimension of admissible deformations of C is the difference in rank $h^0(C, \nu_C) - h^1(C, \nu_C)$.

(1) Using a certain well-known short exact sequence, calculate $h^0(C, \nu_C) - h^1(C, \nu_C)$ to be

$$\int_C c_1(X) + (\dim X - 3)(1 - g).$$

(2) The dimension of the moduli space of complex structures on C is $3g - 3$ by Exercise 1. Identify this term in the above formula.
(3) Read the additional explanations of section $\frac{1}{2}$ and the beginning of section $1\frac{1}{2}$ of [11].

Finally, we get to some calculations.

Exercise 3. Prove rigorously that $N_1 = 1$.

Exercise 4. We consider $d = 2$. The following is the outline of proof that $N_2 = 1$.

(1) The space of conics in $\mathbb{P}^2$ is parametrized by $\mathbb{P}^5$. More precisely, $[a : b : c : d : e : f] \in \mathbb{P}^5$ corresponds to the conic given by

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0.$$

(2) Imposing that the conic $[a : b : c : d : e : f]$ meets $P = [p : q : r]$ translates into

$$ap^2 + bq^2 + cr^2 + dpq + epr + fqr = 0,$$

hence a linear equation on $\mathbb{P}^5$.
(3) We have $3 \times 2 - 1 = 5$ general points in $\mathbb{P}^2$. By a linear coordinate change of $\mathbb{P}^2$, we may assume them to be $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$, $P_3 = [0 : 0 : 1]$, $P_4 = [1 : 1 : 1]$ and $P_5 = [p : q : r]$.
(4) Solving the equations gives a unique solution, hence $N_2 = 1$.

Does this method extend to $d \geq 3$? Why not? The natural generalization of this method is counting what types of curves of degree d passing through how many general points? What's the number of such curves?

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[2] In deformation theory, deformations typically are given by some H^0 and obstructions by some H^1.

[3] Expected here translates into the map *ob* being non-degenerate. More about this in a lecture or another exercise session.

Exercise 5. The problem of calculating all N_d was solved recursively by Kontsevich in the early 90s using his moduli space of stable maps. Read the introduction to that solution in [5, Section 0]. Answer the following questions:

- What does the open part of the moduli space consist of?
- In order to use intersection theory to define invariants, one needs compact moduli spaces. Why is that?
- If one takes limits of (embedded) curves in $\mathbb{P}^2$, 'bad' singularities may occur like cusps. Exercise 8 below illustrates how this is undesirable for curve counts. Explain how the moduli space of stable maps circumvents that problem.

Exercise 6. We are now interested in counting the number of conics that are tangent to 5 general lines. A conic will meet a line in two points. Requiring these two points to come together (shrinking their distance if you will) is a codimension 1 condition in $\mathbb{P}^5$. Doing this 5 times hence yields a zero-dimensional counting problem. Find out the issue in the following argument:

(1) As before the space of conics in $\mathbb{P}^2$ consists of $[a : b : c : d : e : f] \in \mathbb{P}^5$ corresponding to
$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0.$$
(2) Consider a line L. By a linear coordinate change, we may assume that L is given by $y = 0$. Hence the intersection points of $[a : b : c : d : e : f]$ and L are given by
$$ax^2 + cz^2 + exz = 0.$$
This equation has a double root if and only if its discriminant $e^2 - 4ac = 0$.
(3) We conclude that meeting a general line in a point of maximal tangency imposes a degree 2 condition. Hence the solution to our enumerative problem is the intersection of 5 general degree 2 hypersurfaces in $\mathbb{P}^5$ yielding the invariant $2^5 = 32$.
(4) However, the correct answer is 1, we are off by 31! Where is the flaw in the argument?
(5) Why did this problem not occur in Exercise 4?
(6) Advanced: Use excess intersection theory to correct the above curve count. Hint: This problem is treated in [4] and [8].

Exercise 7. The invariant N_3 can be calculated explicitly using a beautiful geometric trick. Consider the complete linear system $L = |O_{\mathbb{P}^2}(3)|$ of degree 3 effective divisors on $\mathbb{P}^2$.

(1) Show that L is of dimension 9.
(2) Show that requiring that elements of L pass through 8 general points yields a pencil of cubics L' whose members may be expressed as
$$ag_1 + bg_2 = 0$$

for $[a:b] \in \mathbb{P}^1$ and smooth cubics $g_1 = 0, g_2 = 0$ passing through the 8 general points.

(3) Provided that the 8 points are chosen generically, the curves in L' are at worst nodal. This follows for example from [9, Proposition 2.1], though there may be some more direct ways of showing it, e.g. by Kodaira's, resp. Beauville's, classification of elliptic surfaces, resp. elliptic fibrations over $\mathbb{P}^1$.
(4) Show all $f \in L'$ meet another 9th point. Conclude that L' has 9 basepoints.
(5) Construct the universal family $S \to \mathbb{P}^1$ of L'. This is an elliptic fibration.
(6) Calculate the Euler characteristic $e(S)$ of S.
(7) Calculate the number of nodal fibers in terms of $e(S)$.
(8) Conclude that $N_3 = 12$.
(9) Why does this argument not generalize to $d \geq 4$?

Exercise 8. This exercise is a variation of the previous one to the non-general case and illustrates how the count of curves changes in non-generic situations. Fix a smooth cubic E in $\mathbb{P}^2$. Let P_0 be a flex point of E and consider the elliptic curve (E, P_0), i.e. we choose P_0 as our zero element. Locally, E can be given in some Weierstrass coordinates

$$E : y^2 = x^3 - ax - b,$$

where P_0 is e.g. the point at infinity. Moreover, choose P to be a point of order 9.

(1) Show that if C is a curve of degree 3 meeting E only at P (of multiplicity 9), then C is irreducible and reduced.
(2) Let L be the linear system consisting of curves of degree 3 meeting E only at P. Show that L is of dimension 1, hence a pencil. Intuitively, requiring for a degree 3 curve C to meet E in only 1 point is a codimension 8 condition. Indeed, each time two intersection points come together, this cuts down the dimension by 1 and we do this 8 times. More rigorously, use the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(3) \to \mathcal{O}_E(9P) \to 0,$$

obtained by noting that $3H|_E \sim 9P$.
(3) Show that in L, there is exactly one curve D that is singular at P.
(4) Show that D is nodal at P.
(5) Construct the universal family $S \to \mathbb{P}^1$ by 9 successive blow ups. Keep track of the total transform of D.
(6) $S \to \mathbb{P}^1$ is an elliptic fibration. Using the same argument as in exercise 7, conclude that the number of rational nodal curves meeting E only at P is 3.
(7) Repeat the previous argument to the case of a general E and curves meeting E only at the flex point P_0. This time, the divisor D is 3

times the line at infinity. Because E was chosen to be general, we may assume that the pencil contains at worst nodal curves. Conclude that the number of rational nodal curves meeting E only at P is 2.

(8) It gets worse. Consider the (special) elliptic curve

$$E_0 : f_0 = y^2z - x^3 - z^3 = 0, P_0 = [0 : 1 : 0]$$

and the pencil

$$L' : af_0 + bz^3 = 0.$$

Using a variation of the previous argument, show that the only rational curve in L' is a cuspidal cubic.

(9) In conclusion, by specializing a general elliptic curve E to E_0, the count of degree 3 rational curves meeting E at a flex point goes from 2 nodal curves to 1 cuspidal curve. In the relevant moduli space, two nodal curves converge to a cuspidal curve.

(10) This exercise is [12, Proposition 1.5].

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[5] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, arXiv:alg-geom/9608011.

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[12] N. Takahashi, *Curves in the complement of a smooth plane cubic whose normalizations are* $\mathbb{A}^1$, arXiv:alg-geom/9605007.