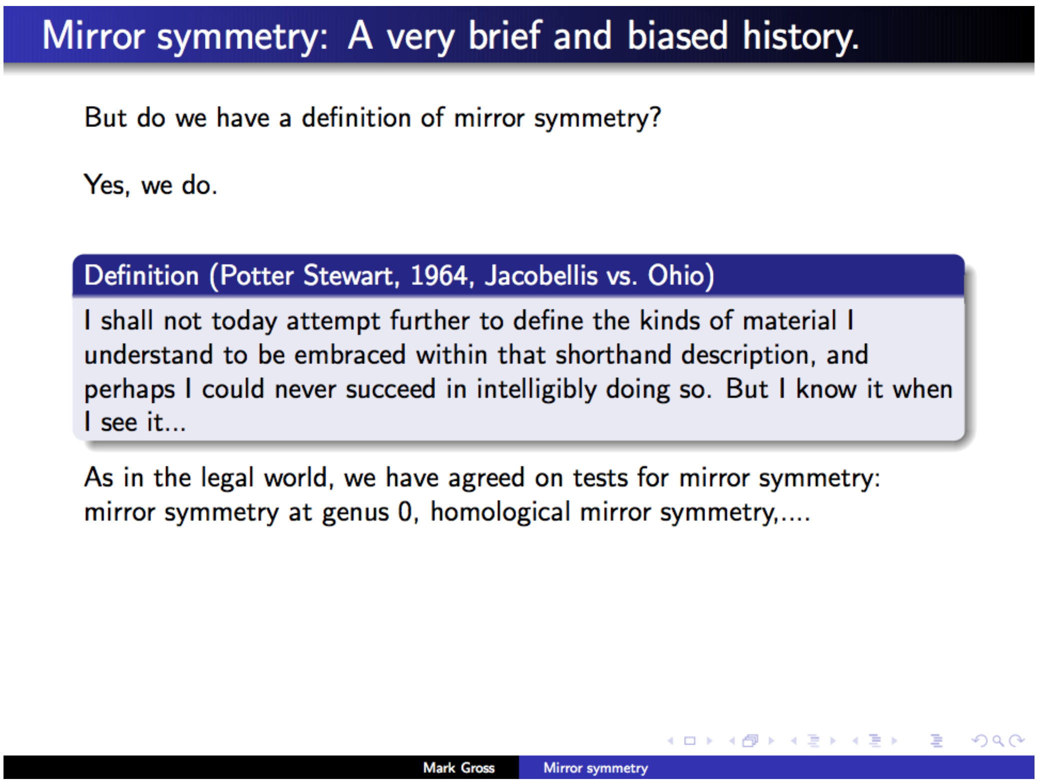
SYZ AND TROPICAL CURVE COUNTS

As a review of the last exercise set, recall Gross' contribution [10, Slide 16] to the definition of mirror symmetry:



Suggested reading

- A little bit of self-propaganda: [8], especially the introduction.
- Any of [9].
- Watch the talk http://media.kias.re.kr/detailPage.do?pro_seq=741&type=p on the KIAS media archive given by P. Overholser with title *Toric degenerations, affine manifolds*.

Exercise 1. In the lecture, we saw a first approximation of the SYZ conjecture in the case where the base B is a vector space. This has a natural extension to the case when B is an affine manifold. In the attached extract from [4] below, go through the construction of semi-flat SYZ mirror symmetry and do the exercises.

Exercise 2. Sometimes special Lagrangian is relaxed to mean that $\operatorname{im}\Omega|_L = e^{i\theta}\operatorname{vol}_L$, for constant θ. Prove that for elliptic curves (and in fact any curve), special Lagrangians are 'straight' lines. Explain what straight means.

Exercise 3. In the lecture, we described a simplified version of mirror symmetry for elliptic curves. Namely, our complex moduli was the positive imaginary line instead of the entire upper half-plane. To generalize it to the entire upper half plane, the issue is that the complex moduli is parametrized by real 2 dimensions, whereas the Kähler parameter is real 1-dimensional. The solution consists in introducing the complexified Kähler form. Use [5] and [16] to extend the elliptic curve SYZ picture to the general case.

Exercise 4. Using [1] as a reference and starting from McLean's theorem, justify why every point of a Calabi-Yau X should be in one and only one special Lagrangian torus.

Exercise 5. In [2, Remark 1.3], the authors argue that their version of local mirror symmetry is complementary to Gross-Siebert. One difference between the two is that [2] uses open Gromov-Witten invariants and the mirror map. Why is that feature not needed in the Gross-Siebert program? Which one is more geometric?

Exercise 6. The purpose of this exercise is to introduce tropical curve counts 'by hand'. Recall that in the previous exercise session, we showed by a geometric trick that the number of degree 3 rational curves N_3 was 12. We will reproduce this calculation tropically. Little to no previous knowledge of tropical geometry is required. We closely follow section 4.2 of [8].

(1) Convince yourself that the number of degree $-K_{\mathbb{P}^2} = 3H$ rational curves in $\mathbb{P}^2$ passing through 8 general points is the same as the number of degree $-K_S = 3H - E_1 - E_2 - E_3$ curves in $S = \operatorname{Bl}_{3\mathrm{pt}}\mathbb{P}^2$ passing through 5 general points. Here the E_i are the classes of the exceptional divisors.
(2) Fig. 2 of [8] gives the fan of the toric blowup $\operatorname{Bl}_{3\mathrm{pt}}\mathbb{P}^2$ with rays generated by primitive vectors ρ_i. Consider the tropical curve depicted in Fig. 3. If you zoom out enough, the curve looks like the fan of $\operatorname{Bl}_{3\mathrm{pt}}\mathbb{P}^2$ with one ray going in each of the directions ρ_i. This means that the degree of the tropical curve is $\Delta = \rho_1 + \cdots + \rho_6$. Given that we are counting curves in class $-K_S$ this is expected.
(3) The combinatorial type of our tropical curves correspond to a subdivision of the Newton polytope of $-K_S$. Find all possible subdivisions and write down a tropical curve of that type.

(4) When will a curve corresponding to a subdivision be reducible?
(5) How do we avoid obtaining reducible curves?
(6) When will a curve be rational?
(7) The starting curve is the tropical curve of Fig. 3. The 5 marked points are distributed on it in general position.
(8) Moduli spaces of tropical curves have a particularly nice structure. Deforming a curve corresponds to changing the length of its edges. In our case, we have a pencil, hence our moduli space is 1-dimensional and locally we are only changing the length of one edge at a time. When a marked point becomes a vertex of the tropical curve, this changes the combinatorial type of the tropical curve and the moduli space branches off into two directions. The moduli space of degree Δ tropical curves in our pencil is the graph of Fig. 4. The strategy of proof is to follow the curves and record each time we encounter a rational tropical curve, which are indicated by stars in Fig. 4.
(9) Follow along the moduli space by observing what happens to the curves in Fig. 5 and 6.
(10) Understand why the last curve is counted with multiplicity 4.
(11) Read section 4.2.2.
(12) Do the exercises 4.13 at the end of section 4.2.

References

[1] D. Auroux, *Special Lagrangian fibrations, mirror symmetry and Calabi-Yau double covers*, arXiv:0803.2734.
[2] K. Chan, S.-C. Lau, N. C. Leung, *SYZ mirror symmetry for toric Calabi-Yau manifolds*, arXiv:1006.3830.
[3] Mirror Symmetry, Clay Mathematics Monographs, Volume 1.
[4] Dirichlet Branes and Mirror Symmetry, Clay Mathematics Monographs, Volume 4.
[5] R. Dijkgraaf *Mirror Symmetry and Elliptic Curves*, http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.128.8694&rep=rep1&type=pdf
[6] W. Fulton, *Intersection Theory*, Springer, 2nd ed., 1998.
[7] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, arXiv:alg-geom/9608011.
[8] M. van Garrel, P. Overholser and H. Ruddat, *Enumerative aspects of the Gross-Siebert program*, arXiv:1410.4783.
[9] M. Gross, *Tropical Geometry and Mirror Symmetry*, CBMS, vol. 114.
[10] M. Gross, *www.dpmms.cam.ac.uk/∼mg475/utah.pdf*.
[11] M. Gross and B. Siebert, *An invitation to toric degenerations*, arXiv:0808.2749.
[12] S. Katz, *Enumerative Geometry and String Theory*, AMS 2006.
[13] M. Kool, V. Shende and R. P. Thomas, *A short proof of the Göttsche conjecture*, arXiv:1010.3211.
[14] N. C. Leung, *Mirror symmetry without corrections*, arXiv:math/0009235.
[15] R. Pandharipande and R. Thomas, *13/2 ways of counting curves*, arXiv:1111.1552.
[16] A. Polishchuk and E. Zaslow, *Categorical Mirror Symmetry: The Elliptic Curve*, arXiv:math/9801119.
[17] N. Takahashi, *Curves in the complement of a smooth plane cubic whose normalizations are* $\mathbb{A}^1$, arXiv:alg-geom/9605007.

us to identify $M_{\mathbb{R}}$ with $N_{\mathbb{R}}$; in fact, $\check{\delta} : M_{\mathbb{R}} \to N_{\mathbb{R}}$ is easily seen to be this identification. This identifies $\Gamma \subseteq M_{\mathbb{R}}$ with

$$\check{\Gamma} = \{d\alpha(\gamma) \in N_{\mathbb{R}} | \gamma \in \Gamma\}.$$

EXERCISE 6.11. Show that in Example 6.7, (2), there is no convex function $K : M_{\mathbb{R}} \to \mathbb{R}$ with the property that $K - K \circ \Psi_\gamma = \alpha(\gamma)$ for some affine linear $\alpha(\gamma)$, for all $\gamma \in \Gamma$. Thus $B = M_{\mathbb{R}}/\Gamma$ cannot be a moduli space of special Lagrangian tori on some Calabi-Yau manifold.

6.2. The semi-flat SYZ picture

We shall now use the structures on the base of a special Lagrangian fibration detailed in the previous section to describe a simple form of mirror symmetry. See [**331**] for more details on semi-flat mirror symmetry.

6.2.1. The basic version. We can now use the structures discussed in §6.1 to define a toy version of mirror symmetry. Fix throughout this section an affine manifold B, and assume that all transition maps are in $M_{\mathbb{R}} \rtimes \mathrm{GL}(n, \mathbb{Z})$ (rather than $\mathrm{Aff}(M_{\mathbb{R}})$).

Given this data, let $y_1, \ldots, y_n$ be local affine coordinates; then one obtains a family of lattices in TB, generated by $\partial/\partial y_1, \ldots, \partial/\partial y_n$. This is well-defined globally because of the integrality assumption: a change of coordinates will produce a different basis for the same lattice, related by an element of $\mathrm{GL}(n, \mathbb{Z})$. This defines a local system $\Lambda \subseteq TB$. Similarly, locally $dy_1, \ldots, dy_n$ generate a local system $\check{\Lambda} \subseteq T^*B$. We will also write $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and $\check{\Lambda}_{\mathbb{R}} = \check{\Lambda} \otimes_{\mathbb{Z}} \mathbb{R}$. Again these are local systems contained in TB and T^*B, but now we allow real linear combinations of $\partial/\partial y_1, \ldots, \partial/\partial y_n$ or $dy_1, \ldots, dy_n$ as local sections.

We can now define two torus bundles:

$$X(B) := TB/\Lambda$$

is a T^n-bundle over B, as is

$$\check{X}(B) := T^*B/\check{\Lambda}.$$

We write

$$f : X(B) \to B$$

and

$$\check{f} : \check{X}(B) \to B,$$

and we say these are dual torus bundles.

These bundles $X(B)$ and $\check{X}(B)$ come along with some additional structures. First, $\check{X}(B)$ is naturally a symplectic manifold: the canonical symplectic form on T^*B descends to a symplectic form ω on $\check{X}(B)$. Second, $X(B)$ carries a complex structure. Locally, this can be described in terms of holomorphic coordinates. Let $U \subseteq B$ be an open set with affine coordinates $y_1, \ldots, y_n$, so TU has coordinate functions $y_1, \ldots, y_n$, $x_1 = dy_1, \ldots, x_n = dy_n$. Then

$$q_j = e^{2\pi\sqrt{-1}(x_j + \sqrt{-1}y_j)}$$

gives a system of holomorphic coordinates on TU/Λ.

Exercise 6.12. Check to see how the coordinates q_j transform under an integral affine change of coordinates y_j. Observe that this change of coordinates is holomorphic.

Completing this exercise shows that these coordinates give a well-defined complex structure on $X(B)$. Note that this can be described in a coordinate-free manner as follows. The differential of the developing map takes $T\tilde{B} = \tilde{B} \times M_{\mathbb{R}}$ to $TM_{\mathbb{R}} = M_{\mathbb{R}} \times M_{\mathbb{R}}$, which allows us to pull back the natural complex structure on $M_{\mathbb{R}} \times M_{\mathbb{R}} = M_{\mathbb{R}} \oplus \sqrt{-1}M_{\mathbb{R}} = M_{\mathbb{R}} \otimes \mathbb{C}$. The complex structure described by the above coordinates is just induced by the pullback of the canonical complex structure on $M_{\mathbb{R}} \otimes \mathbb{C}$.

Note that in addition $X(B)$ comes along with a natural local holomorphic n-form Ω given by

$$\frac{dq_1 \wedge \cdots \wedge dq_n}{q_1 \cdots q_n}.$$

This is not always globally well-defined, although $\Omega \wedge \bar{\Omega}$ is, as we see in the following exercise.

Exercise 6.13. Check that Ω is preserved by a change of coordinates $y_1, \ldots, y_n$ in $M_{\mathbb{R}} \rtimes \mathrm{SL}_n(\mathbb{Z})$, and is only preserved up to sign by a change of coordinates in $M_{\mathbb{R}} \rtimes \mathrm{GL}(n, \mathbb{Z})$. Thus, if the holonomy representation is contained in $M_{\mathbb{R}} \rtimes \mathrm{SL}_n(\mathbb{Z})$, we obtain a global holomorphic n-form Ω.

Now suppose in addition we have a metric g of Hessian form on B, with potential function $K : \tilde{B} \to \mathbb{R}$. Then in fact both $X(B)$ and $\check{X}(B)$ become Kähler manifolds:

Proposition 6.14. *$K \circ f$ is a (multi-valued) Kähler potential on $X(B)$, defining a Kähler form $\omega = 2\sqrt{-1}\partial\bar{\partial}(K \circ f)$. This metric is Ricci-flat if and only if K satisfies the real Monge-Ampère equation*

$$\det \frac{\partial^2 K}{\partial y_i \partial y_j} = \text{constant}.$$

Proof. Working locally with affine coordinates (y_i) and multi-valued complex coordinates $z_i = \frac{1}{2\pi\sqrt{-1}} \log q_i = x_i + \sqrt{-1}y_i$, we compute $\omega = 2\sqrt{-1}\partial\bar{\partial}(K \circ f) = \frac{\sqrt{-1}}{2} \sum \frac{\partial^2 K}{\partial y_i \partial y_j} dz_i \wedge d\bar{z}_j$ which is clearly positive. Furthermore, ω^n is proportional to $\Omega \wedge \bar{\Omega}$ if and only if $\det(\partial^2 K/\partial y_i \partial y_j)$ is constant. $\square$

We write this Kähler manifold as $X(B, K)$.

Dually we have

PROPOSITION 6.15. *In local canonical coordinates $y_i, \check{x}_i$ on T^*B, the functions $z_i = \check{x}_i + \sqrt{-1}\partial K/\partial y_i$ on T^*B induce a well-defined complex structure on $\check{X}(B)$, with respect to which the canonical sympletic form ω is the Kähler form of a metric. Furthermore this metric is Ricci-flat if and only if K satisfies the real Monge-Ampère equation*

$$\det \frac{\partial^2 K}{\partial y_i \partial y_j} = \text{constant}.$$

PROOF. As in Exercise 6.12, it is easy to see that an affine linear change in the coordinates y_i (and hence an appropriate change in the coordinates $\check{x}_i$) results in a linear change of the coordinates z_i, so they induce a well-defined complex structure invariant under $\check{x}_i \mapsto \check{x}_i + 1$, and hence a complex structure on $\check{X}(B)$. So one computes that

$$\omega = \sum d\check{x}_i \wedge dy_i = \frac{\sqrt{-1}}{2} \sum g^{ij} dz_i \wedge d\bar{z}_j$$

where $g_{ij} = \partial^2 K/\partial y_i \partial y_j$. Then the metric is Ricci-flat if and only if $\det(g^{ij}) = \text{constant}$, if and only if $\det(g_{ij}) = \text{constant}$. $\square$

As before, we call this Kähler manifold $\check{X}(B, K)$, and now observe

PROPOSITION 6.16. *There is a canonical isomorphism*

$$X(B, K) \cong \check{X}(\check{B}, \check{K})$$

of Kähler manifolds, where $(\check{B}, \check{K})$ is the Legendre transform of (B, K).

PROOF. Of course $B = \check{B}$ as manifolds, but they carry different affine structures. In addition, the metrics g induced by K and $\check{K}$ coincide by Proposition 6.4. Now identify TB and $T^*B = T^*\check{B}$ using this metric, so in local coordinates (y_i), $\partial/\partial y_i$ is identified with $\sum_j g_{ij} dy_j$. But $d\check{y}_i = \sum \frac{\partial^2 K}{\partial y_i \partial y_j} dy_j = \sum_j g_{ij} dy_j$, so $\partial/\partial y_i$ and $d\check{y}_i$ are identified. Thus this identification descends to a canonical identification of $X(B)$ and $\check{X}(\check{B})$.

We just need to check that this identification gives an isomorphism of Kähler manifolds. But the complex coordinate $\check{z}_i = \check{x}_i + \sqrt{-1}\partial\check{K}/\partial\check{y}_i$ on $T^*\check{B}$ is identified with the coordinate $z_i = x_i + \sqrt{-1}y_i$ on TB under this identification, so the complex structures agree. Finally, the Kähler forms are

$$\tfrac{\sqrt{-1}}{2} \sum g_{ij} dz_i \wedge d\bar{z}_j \text{ and } \tfrac{\sqrt{-1}}{2} \sum \check{g}^{ij} dz_i \wedge d\bar{z}_j$$

respectively, where $\check{g}_{ij} = g(\partial/\partial\check{y}_i, \partial/\partial\check{y}_j)$. But

$$\begin{aligned}\delta_{ik} = \sum_j g_{ij}g^{jk} &= \sum_j g^{jk} g\left(\partial/\partial y_i, \partial/\partial y_j\right)\\ &= \sum_j g^{jk} g(\sum_k g_{ik}\partial/\partial\check{y}_k, \sum_l g_{jl}\partial/\partial\check{y}_l)\\ &= \sum g^{jk} g_{ik}g_{jl}\check{g}_{kl}\\ &= \sum g_{il}\check{g}_{lk},\end{aligned}$$

so $\check{g}^{ij} = g_{ij}$.

Thus the two Kähler forms also agree. □

We can now state more explicitly the simplest form of mirror symmetry.

Definition 6.17. If B is an affine manifold (with transition functions in $\mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{Z})$) then we say $X(B)$ and $\check{X}(B)$ are *SYZ dual.* If, in addition, we have a convex function $K : \tilde{B} \to \mathbb{R}$, then we say

$$X(B, K) \cong \check{X}(\check{B}, \check{K})$$

and

$$\check{X}(B, K) \cong X(\check{B}, \check{K})$$

are SYZ dual.

In the former case, this is a duality between complex and symplectic manifolds, and in the latter between Kähler manifolds. We view either case as a simple version of mirror symmetry.

6.2.2. Semi-flat differential forms. In this section we will discuss forms on both $X(B)$ and $\check{X}(B)$, and their interplay, so it will be useful to work locally with affine coordinates $y_1, \ldots, y_n$ on B, and corresponding coordinates $x_1, \ldots, x_n$ on the tangent bundle and $\check{x}_1, \ldots, \check{x}_n$ on the cotangent bundle. In addition, we will assume in this section that the transition maps of B are contained in $\mathbb{R}^n \rtimes \mathrm{SL}_n(\mathbb{Z})$, so that $X(B)$ carries a nowhere vanishing holomorphic n-form.

Definition 6.18. A *semi-flat differential form* of type (p, q) on $X(B)$ (or $\check{X}(B)$) is a $(p + q)$-form written locally on B as

$$\sum_{\substack{\#I=p\\ \#J=q}} \alpha_{IJ}(y)dy_I \wedge dx_J$$

(or

$$\sum_{\substack{\#I=p\\ \#J=q}} \alpha_{IJ}(y)dy_I \wedge d\check{x}_J$$